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Free-Boundary Problems for Potential and Stokes Flows via Nonsmooth Analysis

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FREE BOUNDARY PROBLEMS FOR POTENTIAL AND STOKES FLOWS VIA NONSMOOTH ANALYSIS*

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Abstract. A new approach to some free boundary problems of the type of jets and cavities for potential flows is introduced. Both potential *and* Stokes flows are considered. The variable domain problems are relaxed so that they become nonsmooth optimization problems on fixed domains for somewhat singular state equations. State equations are considered, and multivalued generalized gradients of the variational functionals are studied. The method is constructive.

Key words. free boundary, Stokes problem, nonsmooth analysis

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1. Introduction.

Consider the following now-classical variational problem introduced and solved by Alt and Caffarelli [2], and studied extensively by Alt, Caffarelli, and Friedman. (See [3] and [8] and references given there.) (See also [14] for numerical considerations; for the simplicity of presentation we discuss the very particular geometry: $\Omega = (-a, a) \times (0, 2)$.)

Find $w \in H^1(\Omega)$ satisfying the boundary conditions

$$w = 0 \text{ in } \{(x, 0); -a < x < a\},\ w = 2 \text{ in } \{(x, 2); -a < x < a\} \tag{1.1.1}$$

such that the variational functional

$$\mathbf{J}(w) = \int_\Omega \left[|\nabla w|^2 + g^2 \mathbf{I}_{\{w>0\}}\right] \tag{1.1.2}$$

is minimized. Here $\mathbf{I}_D$ is a characteristic function of the set D, i.e.,

$$\mathbf{I}_D(x) = \begin{cases} 1 & \text{if } x \in D, \\ 0 & \text{if } x \notin D, \end{cases} \tag{1.1.3}$$

and $g \geq 0$ is a given function.

It is well known (see [2]) that, under certain conditions, a minimizer w satisfies

$$\begin{gathered} \Delta w = 0 \ \text{ in } \ \Omega \cap \{w > 0\}, \\ |\nabla w| = g,\ w = 0 \ \text{ in } \ \Omega \cap \partial\{w > 0\}, \\ w = 0 \text{ in } \{(x, 0); -a < x < a\},\ w = 2 \text{ in } \{(x, 2); -a < x < a\}, \\ w_x = 0 \text{ in } \{(\pm a, y); 0 < y < 2\}. \end{gathered} \tag{1.1.4}$$

REMARK 1.1.1. *Moreover, if $g_y \leq 0$ then, using monotone rearrangemets (see [11]) one can easily show that that there exists a minimizer w such that $w_y \geq 0$. That implies that there exists a function $u = u(x)$ such that $\Omega \cap \partial\{w > 0\} = \{(x, u(x)); -a < x < a\}$. Furthermore, if $g \in C^{k,\alpha}(\Omega)$ then, by the theorem of Alt and Caffarelli, $u \in C^{k+1,\alpha}$.*

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The applications of this problem are mainly in potential fluid mechanics, i.e., in free boundary problems of the type of jets and cavities for potential flows. (See [8] and the references given there.)

Unfortunately, the above variational approach failed, in the case of Stokes and Navier–Stokes equation. The reason is that there is no known analog of the functional (1.1.2).

In this paper we introduce a new approach to this problem. The approach is discussed in the case of potential flow (§2), and in the case of the Stokes flow (§3). Results of the §2 were announced by Stojanovic in [15]. See [4] for a related method.

2. Potential flow.

2.1. Statement of the problem. To further motivate our approach, we observe that the "Euler equation" for the minimizer of the functional (1.1.2) is

$$\begin{gathered}\Delta w=\xi_{\Omega\cap\partial\{w>0\}} \text{ in } \mathcal{D}'(\Omega),\\ w=0 \text{ in } \{(x,0);-a<x<a\},\ w=2 \text{ in } \{(x,2);-a<x<a\},\\ w_x=0 \text{ in } \{(\pm a,y);0<y<2\},\end{gathered}\tag{2.1.1}$$

where for any regular surface Γ, the measure ξ_Γ is defined by

$$\xi_\Gamma(\varphi)=\int_\Gamma g\varphi d\sigma.\tag{2.1.2}$$

Of course, (2.1.1) is a very difficult equation since the measure on the right-hand side depends on a solution. On the other hand, if the right-hand side does not depend on a solution, i.e., if the equation is merely

$$\begin{gathered}\Delta w=\xi_\Gamma \text{ in } \mathcal{D}'(\Omega),\\ w=0 \text{ in } \{(x,0);-a<x<a\},\ w=2 \text{ in } \{(x,2);-a<x<a\},\\ w_x=0 \text{ in } \{(\pm a,y);0<y<2\},\end{gathered}\tag{2.1.3}$$

for some given (fixed) regular surface Γ, then (2.1.3) is a fairly simple equation. So the idea is to study (2.1.3) and then to look for Γ such that if w is the corresponding solution of (2.1.3), then

$$\Gamma=\Omega\cap\partial\{w>0\}.\tag{2.1.4}$$

So, consider the set of admissible shapes (see Remark 1.1.1)

$$U=\left\{u\in H_0^3(-1,1);0\le u(x)\le 1,-1<x<1\right\}.\tag{2.1.5}$$

Denote

$$\Gamma_u=\{(x,u(x));-1<x<1\},\tag{2.1.6}$$

and extend $u\in U$ as zero outside of $(-1,1)$. Define the domain

$$\Omega_u=\{(x,y);\ |x|<a,\ u(x)<y<2\}.\tag{2.1.7}$$

Let $w=w^u$, be the solution of

$$\begin{gathered}\Delta w=0 \text{ in } \Omega_u,\\ w=0 \text{ in } \{(x,0);-a<x<-1 \text{ or } 1<x<a\},\ w=0 \text{ in } \Gamma_u,\\ w=2 \text{ in } \{(x,2);-a<x<a\},\ w_x=0 \text{ in } \{(\pm a,y);0<y<1\}.\end{gathered}\tag{2.1.8}$$

We could also take $a = \infty$ in (2.1.8), i.e., consider a flow in an infinite channel. Then the last condition in (2.1.8) is substituted by the requirement that w is bounded.

In the context of potential fluid mechanics, w is a stream function. If a stream function w is known then, of course, the velocity vector field $\mathbf{v}$ can be computed easily as $\mathbf{v} = \langle w_y, -w_x \rangle$.

The problem we propose is the following.

For given $g = g(x, y)$ such that (we will not always have to assume this much)

$$g \in C^{1,1}(\Omega), \tag{2.1.9}$$

$$g = 0 \text{ in } \Omega \cap \{|x| > 1\}, \tag{2.1.10}$$

find (if possible) $u \in U$ such that, if w^u is the corresponding solution of (2.1.8), then also

$$|\mathbf{v}| = |\nabla w^u| = g \text{ in } \Gamma_u. \tag{2.1.11}$$

We note that by the Bernoulli's law

$$P + \frac{1}{2}|\nabla w^u|^2 = \text{const} \tag{2.1.12}$$

throughout the fluid (here P denotes the pressure). Hence, we see that requesting specific velocity profile on the immersed obstacle is equivalent to requesting the specific pressure (and hence, force) profile. In §3 we shall study the exactly analogous problem for the Stokes equation. So the method introduced here, although not as satisfactory as the variational method of Alt and Caffarelli [2], is applicable to more important equations.

2.2. A relaxation of the problem. Suppose that there exists an $u \in U$ such that corresponding w^u solves (2.1.8) and (2.1.11). We shall say then that u is an *exact shape*. Now, extend w^u from Ω_u to Ω as z^u:

$$z^u = \begin{cases} w^u & \text{on } \Omega_u, \\ 0 & \text{on } \Omega \backslash \Omega_u. \end{cases} \tag{2.2.1}$$

Lemma 2.2.1 follows.

LEMMA 2.2.1. *If $u \in U$ is an exact shape, then $z^u \in H^1(\Omega)$, and it is a solution of the following elliptic boundary value problem (with singular right-hand side):*

$$\begin{gathered} \Delta z^u = \xi_u \textit{ in } \Omega, \\ z^u = 0 \textit{ in } \{(x,0); -a < x < a\},\ z^u = 2 \textit{ in } \{(x,2); -a < x < a\}, \\ (z^u)_x = 0 \textit{ in } \{(\pm a, y); 0 < y < 1\}, \end{gathered} \tag{2.2.2}$$

where $\xi_u \in H^{-1}(\Omega)$ is a measure given by

$$\xi_u(\varphi) = \int_{\Gamma_u} g\varphi d\sigma. \tag{2.2.3}$$

Proof. The proof is obvious, since by elliptic estimates w^u is regular in Ω_u, $z^u \in C^{0,1}(\bar{\Omega})$ (regarding regularity near corners, see the beginning of the proof of the Theorem 2.3.1), and in particular $z^u \in H^1(\Omega)$.

By the trace theorem,

$$|\xi_u(\varphi)| \leq \|g\|_{L^2(\Gamma_u)}\|\varphi\|_{L^2(\Gamma_u)} \leq c_u\|g\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)}. \tag{2.2.4}$$

So, in particular, $\xi_u \in H^{-1}(\Omega)$. Also, since $g \geq 0$, ξ_u is a measure.

Now, more explicitly, (2.2.2) can be written as the following: Find $z \in H^1(\Omega)$ such that

$$z^u = 0 \text{ in } \{(x,0); -a < x < a\},\ z^u = 2 \text{ in } \{(x,2); -a < x < a\} \tag{2.2.5}$$

and

$$-\int_\Omega \nabla z^u \cdot \nabla\varphi = \int_{\Gamma_u} g\varphi d\sigma \tag{2.2.6}$$

for all $\varphi \in H^1(\Omega)$ such that

$$\varphi = 0 \text{ in } \{(x,0); -a < x < a\} \cup \{(x,2); -a < x < a\}. \tag{2.2.7}$$

To check (2.2.6), we note that by the maximum principle, a solution of (2.1.8) is positive. Hence (2.1.11) and the boundary condition in (2.1.8) imply that

$$\frac{\partial w^u}{\partial \nu_u} = -g \text{ in } \Gamma_u, \tag{2.2.8}$$

where ν_u is the exterior unit normal to $\partial\Omega_u$. Hence,

$$\begin{aligned} -\int_\Omega \nabla z^u \cdot \nabla\varphi &= -\int_{\Omega_u} \nabla w^u \cdot \nabla\varphi \\ &= \int_{\Omega_u} (\Delta w^u)\varphi - \int_{\partial\Omega_u} \frac{\partial w^u}{\partial\nu_u}\varphi d\sigma = \int_{\Gamma_u} g\varphi d\sigma, \end{aligned} \tag{2.2.9}$$

which completes the proof of the lemma.

LEMMA 2.2.2. *Let z^u be a solution of* (2.2.5)–(2.2.7). *If it happens that $z^u|_{\Gamma_u} = 0$, then $z^u|_{\Omega_u}$ is a solution of* (2.1.8)–(2.1.11), *i.e., u is an exact shape.*

Proof. In the next section we shall prove that z^u is regular enough so that calculations performed here are legitimate. More precisely, by (2.3.6) below, it suffices to assume that $\varphi \in C_0^1(\Omega)$. We have

$$\begin{aligned} \int_{\Gamma_u} g\varphi d\sigma &= -\int_{\Omega_u} \nabla z^u \cdot \nabla\varphi - \int_{\Omega\setminus\Omega_u} \nabla z^u \cdot \nabla\varphi = \int_{\Omega_u} (\Delta z^u)\varphi \\ &+ \int_{\Omega\setminus\Omega_u} (\Delta z^u)\varphi - \int_{\partial\Omega_u} \frac{\partial z^u}{\partial\nu}\varphi d\sigma - \int_{\partial(\Omega\setminus\Omega_u)} \frac{\partial z^u}{\partial\nu}\varphi d\sigma. \end{aligned} \tag{2.2.10}$$

Let ν be exterior to Ω_u, and let

$$z^{u,\text{int}} \overset{\text{def}}{=} z^u|_{\Omega\setminus\Omega_u},\ z^{u,\text{ext}} \overset{\text{def}}{=} z^u|_{\Omega_u}. \tag{2.2.11}$$

Then (2.2.10) implies that

$$\int_{\Gamma_u} g\varphi d\sigma = \int_{\Gamma_u} \left(\frac{\partial z^{u,\text{int}}}{\partial\nu} - \frac{\partial z^{u,\text{ext}}}{\partial\nu}\right)\varphi d\sigma,\ \forall\varphi \in C_0^1(\Omega). \tag{2.2.12}$$

So,

$$g = \frac{\partial z^{u,\text{int}}}{\partial \nu} - \frac{\partial z^{u,\text{ext}}}{\partial \nu} \text{ on } \Gamma_u. \tag{2.2.13}$$

We observe that (2.2.13) always holds for the solution of (2.2.2).

Now, if $z^u|_{\Gamma_u} = 0$, then $z^u|_{\Omega \setminus \Omega_u} = 0$, so that $(\partial z^{u,\text{int}})/\partial \nu = 0$, and then $g = -(\partial z^{u,\text{ext}})/\partial \nu$ on Γ_u, i.e.,

$$g = |\nabla (z^u|_{\Omega_u})| \text{ on } \Gamma_u, \tag{2.2.14}$$

i.e., (2.1.11) holds.

Lemma 2.2.2 motivates the following.

DEFINITION 2.2.1. *$u^* \in U$ is said to solve the* relaxed free boundary problem *if the corresponding z^u defined by* (2.2.2) *is such that*

$$\Phi(u) = \frac{1}{2} \int_{\Gamma_u} (z^u)^2 d\sigma \tag{2.2.15}$$

is minimized, i.e., $u^* \in U$ is such that

$$\Phi(u^*) = \min_{u \in U} \Phi(u). \tag{2.2.16}$$

Of course, an exact shape is a minimizer, i.e., a solution of (2.2.16). On the other hand, a solution of (2.2.16) is an exact shape, provided an exact shape exists.

We do not consider whether an exact shape exists. Rather, we shall study the relaxed problem introduced in Definition 2.2.1.

2.3. The state equation. It will be convenient to state the regularity theorem for the general boundary value. So let ψ be a given function on Ω such that $z^u = \psi$ on $\partial_0 \Omega \subset \partial \Omega$. We assume that the boundary and ψ are sufficiently regular (see [16] for details; also, we shall give some details in the case of the boundary and boundary values in our case). For any $z \in H^1(\Omega)$, we define $\|z\|_{L^\infty(\partial\Omega)}$ as

$$\|z\|_{L^\infty(\partial\Omega)} \stackrel{\text{def}}{=} \inf \left\{ m \geq 0; -m \leq z \leq m \text{ on } \partial\Omega \text{ in } H^1(\Omega) \right\}, \tag{2.3.1}$$

where inequalities in $H^1(\Omega)$ are defined in, e.g., [16]. Also, we define

$$W^{2,q}_{\{y>0\}-loc}(\Omega) \stackrel{\text{def}}{=} \cap_{\epsilon > 0} W^{2,q}(\Omega \cap \{y > \epsilon\}). \tag{2.3.2}$$

We have the next theorem.

THEOREM 2.3.1. *For any $u \in U$ the state equation* (2.2.2) *has a unique weak solution. Let q be such that $2 \leq q < \infty$. If $g \in W^{1,q}(\Omega)$, then*

$$z^u \in W^{1,q}(\Omega) \cap C^\infty(\bar{\Omega} \setminus \Gamma_u), \tag{2.3.3}$$

and the a priori estimate

$$\|z^u\|_{W^{1,q}(\Omega)} \leq c \left(1 + \|u\|_{C^{0,1}(-1,1)}\right) \left(\|g\|_{W^{1,q}(\Omega)} + \|\psi\|_{W^{1,q}(\Omega)}\right). \tag{2.3.4}$$

holds. If in addition $q > 2$, then

$$\|z^u\|_{L^\infty(\Omega)} \leq c \left(1 + \|u\|_{C^{0,1}(-1,1)}\right) \left(\|g\|_{W^{1,q}(\Omega)} + \|\psi\|_{L^\infty(\partial\Omega)}\right). \tag{2.3.5}$$

Moreover, if $g \in W^{2,q}(\Omega)$, and (2.1.10) holds, then (see (2.2.11))

$$z^{u,\text{ext}} \in W^{2,q}(\Omega_u),\ z^{u,\text{int}} \in W^{2,q}_{\{y>0\}-\text{loc}}(\Omega \setminus \bar{\Omega_u}) \tag{2.3.6}$$

and the a priori estimates

$$\|z^{u,\text{ext}}\|_{W^{2,q}(\Omega_u)} \le c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right) \tag{2.3.7}$$

and

$$\begin{aligned} &\|z^{u,\text{int}}\|_{W^{2,q}((\Omega\setminus\Omega_u)\cap\{y>\epsilon\})} \\ &\le c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right) \end{aligned} \tag{2.3.8}$$

hold.

Proof. Since $\xi_u \in H^{-1}(\Omega)$ existence and uniqueness of a weak solution z^u of (2.2.2) is trivial. Also, since z^u is harmonic in $\Omega\setminus\Gamma_u$, it follows that $z^u \in C^\infty(\bar{\Omega}\setminus\Gamma_u)$. Few words are needed here because of the presence of corners in Ω. To prove regularity of z^u in the neighborhood of corners, say, in the neighborhood of $(-a, 0)$, one can extend z^u in $\{x < -a, 0 < y < 2\}$ as $\widetilde{z^u}$ by the formula

$$\widetilde{z^u}(x,y) \overset{\text{def}}{=} \begin{cases} z^u(-2a - x, y) & \text{if } x < -a, \\ z^u(x,y) & \text{if } x \ge -a. \end{cases} \tag{2.3.9}$$

Then since $\widetilde{z^u}$ is continuous on $\{x = -a\}$ and $\widetilde{z^u}_x = 0$ on $\{x = -a\}$, it is elementary to show that $\widetilde{z^u}$ is harmonic across $\{x = -a\}$. Indeed, let $B_\rho(A) = B_1 \cup (B_\rho(A) \cap \{x = -a\}) \cup B_2 \subset \{0 < y < 2\}$ be a ball centered at $A \in \{x = -a\}$ with radius ρ. Here, $B_1 = B_\rho(A) \cap \{x > -a\}$ and $B_2 = B_\rho(A) \cap \{x < -a\}$. Then,

$$\begin{aligned} \int_{B_\rho(A)} \widetilde{z^u}\Delta\varphi &= \int_{B_1} \widetilde{z^u}\Delta\varphi + \int_{B_2} \widetilde{z^u}\Delta\varphi \\ &= \int_{\{x=-a\}\cap B_\rho(A)} \left[-\varphi_x\widetilde{z^u} + \varphi\widetilde{z^u}_x + \varphi_x\widetilde{z^u} - \varphi\widetilde{z^u}_x\right] dy = 0 \end{aligned} \tag{2.3.10}$$

for all $\varphi \in C_0^\infty(B_\rho(A))$, so that $\widetilde{z^u}$ is harmonic across $\{x = -a\}$ as claimed. Henceforth $\widetilde{z^u}$ is as regular in the neighborhood of $(-a, 0)$ as the (extended) boundary data is. In our case the boundary data is $\psi = 0$, so that (2.3.3) follows.

Set $\varphi = \psi - z^u$ in (2.2.6). It easily follows that

$$\begin{aligned} \int_\Omega |\nabla z^u|^2 &= \int_{\Gamma_u} g(\psi - z^u)d\sigma - \int_\Omega \nabla z^u \cdot \nabla\psi \\ &\le \left(\int_{\Gamma_u} g^2 d\sigma\right)^{\frac{1}{2}} \left[\left(\int_{\Gamma_u} \psi^2 d\sigma\right)^{\frac{1}{2}} + \left(\int_{\Gamma_u} (z^u)^2 d\sigma\right)^{\frac{1}{2}}\right] \\ &\quad + \left|\int_\Omega \nabla z^u \cdot \nabla\psi\right|. \end{aligned} \tag{2.3.11}$$

Now since $z^u = (z^u - \psi) + \psi$, using Poincaré inequality, we have

$$\begin{aligned} \|z^u\|_{H^1(\Omega)} &\le c\left(\|\nabla(z^u - \psi)\|_{L^2(\Omega)} + \|\psi\|_{H^1(\Omega)}\right) \\ &\le c\left(\|\nabla z^u\|_{L^2(\Omega)} + \|\psi\|_{H^1(\Omega)}\right). \end{aligned} \tag{2.3.12}$$

Combining (2.3.11) and (2.3.12), we get

$$\begin{aligned}\|z^u\|^2_{H^1(\Omega)} \le c\left(1+\|u\|_{C^{0,1}(-1,1)}\right)&\left[\|g\|_{H^1(\Omega)}\left(\|\psi\|_{H^1(\Omega)}+\|z^u\|_{H^1(\Omega)}\right)\right]\\ &+c\|z^u\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)}+c\|\psi\|^2_{H^1(\Omega)}.\end{aligned} \tag{2.3.13}$$

In (2.3.13) the inequality follows from the proof of the trace theorem (see, e.g., [12] or [7]). Indeed, one can see ([7], p. 132) that for $1 \le q < \infty$, one has

$$\|z^u\|^q_{L^q(\Gamma_u)} \le c\left(1+\|u\|^2_{C^{0,1}(-1,1)}\right)^{\frac{1}{2}}\|z^u\|^q_{W^{1,q}(\Omega)}, \tag{2.3.14}$$

which implies

$$\|z^u\|_{L^q(\Gamma_u)} \le c\left(1+\|u\|_{C^{0,1}(-1,1)}\right)^{\frac{1}{q}}\|z^u\|_{W^{1,q}(\Omega)}. \tag{2.3.15}$$

From (2.3.13) we easily conclude that (2.3.4) holds for $q=2$.

Proceeding, we assume $\frac{1}{q}+\frac{1}{q'}=1$ and

$$\begin{aligned}|\xi_u(\varphi)| &\le \|g\|_{L^q(\Gamma_u)}\|\varphi\|_{L^{q'}(\Gamma_u)}\\ &\le c\left(1+\|u\|_{C^{0,1}(-1,1)}\right)^{\frac{1}{q}}\|g\|_{W^{1,q}(\Omega)}\left(1+\|u\|_{C^{0,1}(-1,1)}\right)^{\frac{1}{q'}}\|\varphi\|_{W^{1,q'}(\Omega)}\\ &= c\left(1+\|u\|_{C^{0,1}(-1,1)}\right)\|g\|_{W^{1,q}(\Omega)}\|\varphi\|_{W^{1,q'}(\Omega)}.\end{aligned} \tag{2.3.16}$$

So, $\xi_u \in (W^{1,q'}(\Omega))^*$ (here X^* represents the dual space of the space X) and

$$\|\xi_u\|_{\left(W^{1,q'}(\Omega)\right)^*} \le c\left(1+\|u\|_{C^{0,1}(-1,1)}\right)\|g\|_{W^{1,q}(\Omega)}. \tag{2.3.17}$$

We know (see, e.g., [1]) that ξ has a representation $\xi(\varphi)=\int_\Omega[f_0\varphi+f_1\varphi_x+f_2\varphi_y]$, for some $f_i \in L^q(\Omega), i=0,1,2$, and

$$\|\xi_u\|_{\left(W^{1,q'}(\Omega)\right)^*} = \sum_{i=0}^{2}\|f_i\|_{L^q(\Omega)}. \tag{2.3.18}$$

Now from elliptic regularity (see [16], p. 179), we have

$$\|z^u\|_{W^{1,q}(\Omega)} \le c\left(\sum_{i=0}^{2}\|f_i\|_{L^q(\Omega)}+\|\psi\|_{W^{1,q}(\Omega)}+\|z^u\|_{H^1(\Omega)}\right). \tag{2.3.19}$$

From (2.3.17)–(2.3.19) and since (2.3.4) is already proved in the case $q=2$, we conclude that (2.3.4) holds.

To prove (2.3.5), we recall (see, e.g., [16], p. 103) that if $q>2$ and if $z^u \le 0$ on $\partial_0\Omega$ in the sense of $H^1(\Omega)$, then

$$\operatorname*{ess\,sup}_{\Omega} z^u \le c\left(\sum_{i=0}^{2}\|f_i\|_{L^q(\Omega)}+\|z^u\|_{L^2(\Omega)}\right). \tag{2.3.20}$$

Hence,

$$\begin{aligned}&\operatorname*{ess\,sup}_{\Omega}\left(z^u-\|z^u\|_{L^\infty(\partial\Omega)}\right)\\ &\le c\left(\sum_{i=0}^{2}\|f_i\|_{L^q(\Omega)}+\|z^u\|_{L^2(\Omega)}+\|z^u\|_{L^\infty(\partial\Omega)}\right),\end{aligned} \tag{2.3.21}$$

and similarly for $-z^u + \|z^u\|_{L^\infty(\partial\Omega)}$. This easily implies (2.3.5).

Now, we shall consider further regularity of $z^u|_{\Omega_u}$ and $z^u|_{\Omega\setminus\Omega_u}$. Since the singular set is on Γ_u, we expect higher regularity in the tangential direction. To prove that this is the case we flatten the Γ_u first, since then it is easier to differentiate.

Define v, $\tilde{g}$, and $\tilde{\varphi}$ by $v(x,y) = z^u(x, y+u(x)), \tilde{g}(x,y) = g(x, y+u(x))\sqrt{1+u'^2(x)}$, $\tilde{\varphi}(x,y) = \varphi(x, y+u(x))$ and operator L by $Lv = \Delta v + v_{yy}(u_x)^2 - 2v_{xy}u_x - v_y u_{xx}$. Of course, L is uniformly elliptic, since the matrix

$$[l_{ij}] = \begin{bmatrix} 1 & -u_x \\ -u_x & 1+u_x^2 \end{bmatrix} \tag{2.3.22}$$

is positive definite. Indeed, $l_{ij}\xi_i\xi_j = (\xi_1 - u_x\xi_2)^2 + \xi_2^2$. So, if c is such that $|u_x| \le c$, then if $|\xi_2| < \frac{1}{2c}|\xi_1|$ then $(\xi_1 - u_x\xi_2)^2 > \frac{1}{4}\xi_1^2$. On the other hand, if $|\xi_2| \ge \frac{1}{2c}|\xi_1|$ then $\xi_2^2 \ge \frac{1}{4c^2}\xi_1^2$. So, it is easy to see that if we take $\alpha = \min\left(\frac{1}{4}, \frac{1}{8c^2}\right)$ then $l_{ij}\xi_i\xi_j \ge \alpha|\xi|^2$.

Also let Ξ_u be the map with the image Ω given by the formula

$$\Xi_u(x,y) = (x, y+u(x)). \tag{2.3.23}$$

Then, $\Delta z^u \circ \Xi_u = Lv$, and since $|\det D\Xi_u| = 1$ (here $D\Xi_u$ is the gradient matrix of the map Ξ_u so that $|\det D\Xi_u|$ is the Jacobian)

$$(Lv)(\tilde{\varphi}) = (\Delta z^u)(\varphi). \tag{2.3.24}$$

Hence

$$\begin{aligned}(Lv)(\tilde{\varphi}) &= \int_{\Gamma_u} g\varphi d\sigma = \int g(x,u(x))\varphi(x,u(x))\sqrt{1+u'^2(x)}dx \\ &= \int_{\{y=0\}} \tilde{g}\tilde{\varphi}dx \stackrel{\text{def}}{=} \tilde{\xi}(\tilde{\varphi}).\end{aligned} \tag{2.3.25}$$

So

$$Lv = \tilde{\xi} \tag{2.3.26}$$

in the sense of distributions. Since the singular set is now on $\{y=0\}$, we expect higher regularity in x direction. To prove that, we want to differentiate (or more precisely, difference) equation (2.3.26) with respect to x. Somewhat more precisely, define the standard difference operator (in the x direction) δ_h^1 as

$$\left(\delta_h^1 u\right)(x) = \frac{1}{h}\left(u(x+h,y) - u(x,y)\right),\ h \ne 0. \tag{2.3.27}$$

Then from (2.3.26) we get

$$(Lv)\left(\delta_{-h}^1\tilde{\varphi}\right) = \tilde{\xi}\left(\delta_{-h}^1\tilde{\varphi}\right). \tag{2.3.28}$$

We shall discuss in some details only the right-hand side. We have

$$\begin{aligned}\tilde{\xi}\ \left(\delta_{-h}^1\tilde{\varphi}\right) &= \int_{\{y=0\}} \tilde{g}\delta_{-h}^1\tilde{\varphi}dx \\ &= -\int_{\{y=0\}} \left(\delta_h^1\tilde{g}\right)\tilde{\varphi}dx \longrightarrow -\int_{\{y=0\}} \tilde{g}_x\tilde{\varphi}dx,\end{aligned} \tag{2.3.29}$$

as $h \to 0$. We conclude that $(Lv)_x(\tilde{\varphi}) = \tilde{\xi}_x(\tilde{\varphi})$, and hence

$$L_1 v_x = \tilde{\xi}_x - v_{yy} 2u_x u_{xx} + v_y u_{xxx}, \tag{2.3.30}$$

where $L_1 w = \Delta w + (u_x)^2 w_{yy} - 2u_x w_{xy} - 3u_{xx} w_y$, and where $\tilde{\xi}_x(\tilde{\varphi}) \stackrel{\text{def}}{=} \int_{\{y=0\}} \tilde{g}_x \tilde{\varphi} dx$. We observe that the differencing performed above is legitimate, since

$$\tilde{\xi}_x - v_{yy} 2u_x u_{xx} + v_y u_{xxx} \in \left(W^{1,q'}\right)^*. \tag{2.3.31}$$

Indeed, $g \in W^{2,q}(\Omega)$; also, observe that $u_{xxx} \in L^2$ and that $L^2 \hookrightarrow \left(W^{1,q'}\right)^*$. Also, since L_1 has the same principal part as L, L_1 is uniformly elliptic as well.

Now we can conclude from (2.3.30)–(2.3.31) that $v_x \in W^{1,q}$. This implies, by the trace theorem, that $v_x|_{\{y=0\}} \in W^{1-\frac{1}{q},q}$, so that

$$v|_{\{y=0\}} \in W^{2-\frac{1}{q},q}. \tag{2.3.32}$$

We observe that because of (2.1.5) and (2.1.10), the preceding analysis is true also in the $\{y > 0\}$ neighborhood of (the preimage of) $(\pm 1, 0)$, so that (2.3.32) holds up to the initial and terminal points of (the preimage of) Γ_u. Elliptic regularity then yields $v|_{\{y \geq 0\}} \in W^{2,q}$.

Unfortunately, we cannot claim the same global result for $v|_{\{y \leq 0\}}$ because of the nonsmoothness of $\partial(\Omega \setminus \Omega_u)$, i.e., we have to localize in $\{y < 0\}$. This concludes the proof of (2.3.6). Now, regarding estimates (2.3.7) and (2.3.8), we have

$$\begin{aligned} \|z^{u,\text{ext}}\|_{W^{2,q}(\Omega_u)} &\leq c\left(\|u\|_{H^3(-1,1)}\right) \|v|_{\{y\geq 0\}}\|_{W^{2,q}(\Xi_u^{-1}(\Omega_u))} \\ &\leq c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right), \end{aligned} \tag{2.3.33}$$

and similarly (after localization in $\{y < 0\}$) for $z^{u,\text{int}}$, which completes the proof of the theorem.

COROLLARY 2.3.1. *If $g \in W^{2,q}(\Omega)$ for some $q > 2$, and if (2.1.10) holds, then $z^u \in C^{0,1}_{\{y>0\}-\text{loc}}(\bar{\Omega})$ and the following a priori estimate holds:*

$$\|z^u\|_{C^{0,1}(\bar{\Omega}\cap\{y\geq\epsilon\})} \leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right), \tag{2.3.34}$$

for any $\epsilon > 0$.

Proof. From (2.3.7) and (2.3.8) and by the imbedding theorem (see, e.g., [9]), we have

$$\begin{aligned} &\|z^{u,\text{ext}}\|_{C^1(\bar{\Omega}_u)} + \|z^{u,\text{int}}\|_{C^1(\overline{\Omega\setminus\Omega_u}\cap\{y\geq 0\})} \\ &\leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right). \end{aligned} \tag{2.3.35}$$

This implies (2.3.34).

COROLLARY 2.3.2. *If $g \in W^{2,q}(\Omega)$ for some $q > 2$, and if (2.1.10) holds, then*

$$z_y^{u,\text{int}} \in C^{0,1-\frac{2}{q}}\left(\overline{\Gamma_u}\right), \tag{2.3.36}$$

and the following a priori estimate

$$\|z_y^{u,\text{int}}\|_{C^{0,1-\frac{2}{q}}(\overline{\Gamma_u})} \leq c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right) \tag{2.3.37}$$

holds.

The interest in this corollary is due to the lack of $(\Omega \setminus \Omega_u)$-global regularity of z^u.

Proof. Let τ and ν be unit tangent and unit normal to Γ_u. More precisely, set $\tau = 1/(\sqrt{1+u'^2})\,\langle 1, u'\rangle$, and $\nu = 1/(\sqrt{1+u'^2})\,\langle u', -1\rangle$. It is elementary to compute that then $z_y^{u,\mathrm{int}} = u'/(\sqrt{1+u'^2}) z_\tau^{u,\mathrm{int}} - 1/(\sqrt{1+u'^2}) z_\nu^{u,\mathrm{int}}$. Since, by Theorem 2.3.1, $z_\tau^{u,\mathrm{int}} = z_\tau^{u,\mathrm{ext}}$ and (also, by Lemma 2.2.2) $z_\nu^{u,\mathrm{int}} = g + z_\nu^{u,\mathrm{ext}}$ on Γ_u, we have

$$z_y^{u,\mathrm{int}}\Big|_{\Gamma_u} = \left(\frac{u'}{\sqrt{1+u'^2}} z_\tau^{u,\mathrm{ext}} - \frac{1}{\sqrt{1+u'^2}}\left[g + z_\nu^{u,\mathrm{ext}}\right]\right)\Big|_{\Gamma_u}. \tag{2.3.38}$$

The corollary follows due to the Ω_u-*global* regularity of $z^{u,\mathrm{ext}}$, and by the imbedding theorem. Indeed,

$$\begin{aligned}
\|z_y^{u,\mathrm{int}}\|_{C^{0,1-\frac{2}{q}}(\overline{\Gamma_u})} &= \left\|\frac{u'}{\sqrt{1+u'^2}} z_\tau^{u,\mathrm{ext}} - \frac{1}{\sqrt{1+u'^2}}\left[g + z_\nu^{u,\mathrm{ext}}\right]\right\|_{C^{0,1-\frac{2}{q}}(\overline{\Gamma_u})} \\
&\le c\|u\|_{H^3(-1,1)}\left[\|z^{u,\mathrm{ext}}\|_{C^{1,1-\frac{2}{q}}(\overline{\Gamma_u})} + \|g\|_{C^{0,1-\frac{2}{q}}(\overline{\Gamma_u})}\right] \\
&\le c\|u\|_{H^3(-1,1)}\left[\|z^{u,\mathrm{ext}}\|_{W^{2,q}(\Omega_u)} + \|g\|_{W^{1,q}(\Omega_u)}\right] \\
&\le c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right).
\end{aligned} \tag{2.3.39}$$

We finish this section with the consideration of existence of a minimizer. Now, in order to claim existence of a minimizer, i.e., existence of a solution of the relaxed problem, one needs compactness. One way of introducing compactness would be to bound the set of admissible shapes to

$$U_b = \{u \in U; \|u\|_{H^3(-1,1)} \le b\}, \tag{2.3.40}$$

where b is some prescribed (large) positive constant.

PROPOSITION 2.3.1. *Let $g \in W^{1,q}(\Omega)$, for some $q > 2$. Then, there exists an $u^* \in U_b$ such that*

$$\Phi(u^*) = \min_{u \in U_b} \Phi(u). \tag{2.3.41}$$

Proof. Let $(u_n)_{n=1,2,\ldots} \subset U_b$ be a minimizing sequence. By Theorem 2.3.1 we know that

$$\|z^{u_n}\|_{H^1(\Omega)} + \|z^{u_n}\|_{C^{1-\frac{2}{q}}(\bar{\Omega})} \le c. \tag{2.3.42}$$

By taking subsequences, if necessary, we can assume without loss of generality that there exist $u^* \in U_b$ and $z^* \in H^1(\Omega)$ such that

$$u^n \to u^* \text{ in } H^2(-1,1), \tag{2.3.43}$$

$$z^{u_n} \rightharpoonup z^* \text{ weakly in } H^1(\Omega),\ z^{u_n} \to z^* \text{ in } C^0(\bar{\Omega}). \tag{2.3.44}$$

Recall that

$$-\int_\Omega \nabla z^{u_n} \cdot \nabla \varphi = \int_{\Gamma_{u_n}} g\varphi d\sigma \tag{2.3.45}$$

for all $\varphi \in H^1(\Omega)$ such that $\varphi|_{\{y=0\}} = \varphi|_{\{y=2\}} = 0$. If, in addition, $\varphi \in C^1(\bar{\Omega})$ then it is easy to see that

$$\lim_{n\to\infty} \int_{\Gamma_{u_n}} g\varphi d\sigma = \int_{\Gamma_{u^*}} g\varphi d\sigma. \tag{2.3.46}$$

Hence, for such φ we can pass $n \to \infty$ in (2.3.45) to conclude

$$-\int_\Omega \nabla z^* \cdot \nabla\varphi = \int_{\Gamma_{u^*}} g\varphi d\sigma \tag{2.3.47}$$

for any $\varphi \in C^1(\bar{\Omega})$ such that $\varphi|_{\{y=0\}} = \varphi|_{\{y=2\}} = 0$. But then, by the density, (2.3.47) holds for all $\varphi \in H^1(\Omega)$ such that $\varphi|_{\{y=0\}} = \varphi|_{\{y=2\}} = 0$. We conclude, by uniqueness, that $z^* = z^{u^*}$. Now since $\Phi(u_n) = \frac{1}{2}\int_{\Gamma_{u_n}}(z^{u_n})^2 d\sigma$, (2.3.43) and (2.3.44) imply that $\lim_{n\to\infty}\Phi(u_n) = \Phi(u^*)$, which completes the proof of the proposition.

2.4. Differentiability properties of the variational functional Φ. Since our problem is to minimize functional Φ, we want to derive information about the *multivalued* generalized gradient of Φ (see also Remark 2.4.1).

To make our results more precise, we shall introduce several definitions.

Let Φ be a real-valued function on the subset U of the Banach space X.

DEFINITION 2.4.1. *Φ is said to be* directionally differentiable *at $u \in U$ if the limit*

$$\lim_{\lambda\downarrow 0}\frac{\Phi(u+\lambda v) - \Phi(u)}{\lambda} \tag{2.4.1}$$

exists for any $v \in X$ such that $u + \lambda v \in U$, for small enough $\lambda > 0$. If that is the case, then the limit in (2.4.1) *is called* directional derivative *and it is denoted by $\Phi'(u; v)$.*

DEFINITION 2.4.2. *Φ is said to be* subdifferentiable *at u, if there exists an $f \in X^*$ such that*

$$\Phi'(u; v) \geq f(v) \tag{2.4.2}$$

for every $v \in X$ such that $u + \lambda v \in U$, for small enough $\lambda > 0$. Set of all such f's is called subdifferential, *and it is denoted by $\partial_*\Phi(u)$.*

DEFINITION 2.4.3. *Φ is said to be* superdifferentiable *at u, if there exists an $f \in X^*$ such that*

$$\Phi'(u; v) \leq f(v) \tag{2.4.3}$$

for every $v \in X$ such that $u + \lambda v \in U$, for small enough $\lambda > 0$. Set of all such f's is called superdifferential, *and it is denoted by $\partial^*\Phi(u)$.* If Φ is both sub- and superdifferentiable at $u \in \mathrm{int}(U)$, and moreover $\partial_*\Phi(u) \cap \partial^*\Phi(u) \neq \emptyset$, then $\partial_*\Phi(u) \cap \partial^*\Phi(u)$ is a singleton and Φ is Gâteaux differentiable.

We go back now to our problem. Of course, $X = H_0^3(-1,1)$, U is defined in (2.1.5).

Proceeding, define the adjoint variable p^u, as a solution of the (adjoint) equation

$$\begin{aligned} \Delta p^u &= \eta_u \text{ in } \Omega, \\ p^u &= 0 \text{ in } \{(x,0); -a < x < a\} \cup \{(x,2); -a < x < a\}, \\ p_x^u &= 0 \text{ in } \{(\pm a, y); 0 < y < 2\}, \end{aligned} \tag{2.4.4}$$

where $\eta_u \in H^{-1}(\Omega)$ is a (signed) measure given by

$$\eta_u(\varphi) = \int_{\Gamma_u} z^u \varphi d\sigma. \tag{2.4.5}$$

Obviously, (2.4.4) is the same type of equation as (2.2.2).

In this section, as before, $z^{u,\mathrm{ext}} = z^u|_{\Omega_u}$ and $z^{u,\mathrm{int}} = z^u|_{\Omega\setminus\Omega_u}$; also, later we shall use the notation $p^{u,\mathrm{ext}} = p^u|_{\Omega_u}$ and $p^{u,\mathrm{int}} = p^u|_{\Omega\setminus\Omega_u}$. That is essential in this calculation, since z^u and p^u are *not* differentiable *across* the Γ_u.

LEMMA 2.4.1. *Let $g \in W^{2,q}(\Omega)$, for some $q \geq 2$. Then*

$$p^{u,\mathrm{ext}} \in W^{2,q}(\Omega_u),\ p^{u,\mathrm{int}} \in W^{2,q}_{\{y>0\}-loc}(\Omega \setminus \bar{\Omega_u}) \tag{2.4.6}$$

and the a priori estimates

$$\|p^{u,\mathrm{ext}}\|_{W^{2,q}(\Omega_u)} \leq c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right), \tag{2.4.7}$$

and

$$\begin{aligned} &\|p^{u,\mathrm{int}}\|_{W^{2,q}((\Omega\setminus\Omega_u)\cap\{y\geq\epsilon\})} \\ &\quad \leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right) \end{aligned} \tag{2.4.8}$$

hold.

Proof. Comparing (2.2.2) and (2.4.4) we see that the only difference is in right-hand sides. Namely, in (2.4.5), $z^u \notin W^{2,q}(\Omega)$. Nevertheless, for example, $z^{u,\mathrm{ext}} \in W^{2,q}(\Omega_u)$, and since η_u depends on z^u only through the trace on Γ_u, and since z^u and $z^{u,\mathrm{ext}}$ have same traces on Γ_u we easily conclude the proof of the lemma.

We shall use the usual notation: $v^+ \stackrel{\mathrm{def}}{=} v\mathbf{I}_{\{v>0\}}$, and $v^- \stackrel{\mathrm{def}}{=} -v\mathbf{I}_{\{v<0\}}$. So, $v = v^+ - v^-$.

Now we are ready to state the following.

THEOREM 2.4.1. *Let $g \in W^{2,q}(\Omega)$, for some $q > 2$. Then Φ is directionally differentiable at any $u \in U$ such that $u(x) > 0$ for $-1 < x < 1$, and*

$$\begin{aligned} &\Phi'(u; v) \\ &= \int_{-1}^{1} \left(z^u (z_y^{u,\mathrm{ext}} v^+ - z_y^{u,\mathrm{int}} v^-)\sqrt{1+u'^2} + \frac{1}{2}(z^u)^2 \frac{u'v'}{\sqrt{1+u'^2}} \right) dx \\ &\quad + \int_{\Gamma_u} \left((gp^{u,\mathrm{ext}})_y v^+ - (gp^{u,\mathrm{int}})_y v^-\right) d\sigma + \int_{-1}^{1} gp^u \frac{u'v'}{\sqrt{1+u'^2}} dx. \end{aligned} \tag{2.4.9}$$

Moreover, if

$$z^u z_y^{u,\mathrm{int}} + (gp^{u,\mathrm{int}})_y \leq z^u z_y^{u,\mathrm{ext}} + (gp^{u,\mathrm{ext}})_y \ \ a.e.\ in\ (-1,1), \tag{2.4.10}$$

then Φ is subdifferentiable at u and

$$\begin{aligned} &\partial_* \Phi(u) \\ &= \left[\left(z^u z_y^{u,\mathrm{int}} + (gp^{u,\mathrm{int}})_y\right)\sqrt{1+u'^2}, \left(z^u z_y^{u,\mathrm{ext}} + (gp^{u,\mathrm{ext}})_y\right)\sqrt{1+u'^2}\right] \\ &\quad - \left(\frac{u'}{\sqrt{1+u'^2}}\left(\frac{1}{2}(z^u)^2 + gp^u\right)\right)' \\ &\stackrel{\mathrm{def}}{=} [l\partial_*\Phi(u), r\partial_*\Phi(u)] \subset L^\infty(-1,1). \end{aligned} \tag{2.4.11}$$

On the other hand, if

$$z^u z_y^{u,\mathrm{int}} + (gp^{u,\mathrm{int}})_y \geq z^u z_y^{u,\mathrm{ext}} + (gp^{u,\mathrm{ext}})_y \ \ a.e.\ in\ (-1,1), \tag{2.4.12}$$

then Φ is superdifferentiable at u and

$$\begin{aligned}
&\partial^*\Phi(u)\\
&= \left[\left(z^u z_y^{u,\mathrm{ext}} + (gp^{u,\mathrm{ext}})_y\right)\sqrt{1+u'^2}, \left(z^u z_y^{u,\mathrm{int}} + (gp^{u,\mathrm{int}})_y\right)\sqrt{1+u'^2}\right]\\
&\quad - \left(\frac{u'}{\sqrt{1+u'^2}}\left(\frac{1}{2}(z^u)^2 + gp^u\right)\right)'\\
&\overset{\mathrm{def}}{=} [l\partial^*\Phi(u), r\partial^*\Phi(u)] \subset L^\infty(-1,1). \qquad (2.4.13)
\end{aligned}$$

Proof. We attempt to differentiate Φ. To this end, for given $u \in U$ and a suitable direction $v \in H_0^3(-1,1)$ (suitable in a sense that $u + \lambda v \in U$ for small enough $\lambda > 0$) we try to compute the (one-sided) directional derivative $\Phi'(u;v)$. Using the regularity result (Theorem 2.3.1, and Corollary 2.3.2), we compute

$$\begin{aligned}
\Phi'(u;v) &= \lim_{\lambda\downarrow 0}\frac{\Phi(u+\lambda v) - \Phi(u)}{\lambda}\\
&= \lim_{\lambda\downarrow 0}\frac{1}{2\lambda}\left(\int_{\Gamma_{u+\lambda v}}(z^{u+\lambda v})^2 d\sigma - \int_{\Gamma_u}(z^u)^2 d\sigma\right)\\
&= \int_{-1}^{1}\left(z^u(z_y^{u,\mathrm{ext}}v^+ - z_y^{u,\mathrm{int}}v^-)\sqrt{1+u'^2} + \frac{1}{2}(z^u)^2\frac{u'v'}{\sqrt{1+u'^2}}\right)dx\\
&\quad + \lim_{\lambda\downarrow 0}\frac{1}{2\lambda}\int_{\Gamma_u}\left((z^{u+\lambda v})^2 - (z^u)^2\right)d\sigma, \qquad (2.4.14)
\end{aligned}$$

Before proceeding with the proof, we shall need the following lemma (more precisely, its corollary).

LEMMA 2.4.2. *Under previous assumptions on u, and v, and for any $\alpha < 1$ the following estimate holds*:

$$\|z^{u+\lambda v} - z^u\|_{C^0(\bar\Omega)} \le c\lambda^\alpha. \qquad (2.4.15)$$

Proof. We need to compare $z^{u+\lambda v}$ and z^u. This is difficult to do in the original domain Ω since the (singular) right-hand sides of the equations that they satisfy act on disjoint sets, so that there is no obvious cancellation. So, the idea of the proof is to map the original domain into different domains in such a way that the cancellation takes place.

As before, let Ξ_u be the map with the image Ω given by the formula $\Xi_u(x,y) = (x, y+u(x))$. Then $\Xi_u^{-1}(x,y) = (x, y-u(x))$, and (set $A = (x,y)$) $\mathrm{dist}(\Xi_{u+\lambda v}^{-1}(A) - \Xi_u^{-1}(A)) \le c\lambda$. Now consider $\tilde z^{u+\lambda v}$ and $\tilde z^u$ defined as

$$\tilde z^{u+\lambda v} = z^{u+\lambda v}\circ\Xi_{u+\lambda v},\ \tilde z^u = z^u\circ\Xi_u, \qquad (2.4.16)$$

and operators L_u and $L_{u+\lambda v}$ defined by

$$L_u w = \Delta w + w_{yy}(u_x)^2 - 2w_{xy}u_x - w_y u_{xx}, \qquad (2.4.17)$$

$$\begin{aligned}
L_{u+\lambda v}w &= \Delta w + w_{yy}(u_x+\lambda v_x)^2 - 2w_{xy}(u_x+\lambda v_x) - w_y(u_{xx}+\lambda v_x)\\
&= L_u w + \lambda\left[w_{yy}(2u_x v_x + \lambda v_x^2) - 2w_{xy}v_x - w_y v_{xx}\right]. \qquad (2.4.18)
\end{aligned}$$

Then $\tilde{z}^{u+\lambda v} - \tilde{z}^u$ satisfies the equation

$$\begin{aligned} & L_u\left(\tilde{z}^{u+\lambda v} - \tilde{z}^u\right) \\ & = \gamma - \lambda\left[\tilde{z}_{yy}^{u+\lambda v}(2u_x v_x + \lambda v_x^2) - 2\tilde{z}_{xy}^{u+\lambda v} v_x - \tilde{z}_y^{u+\lambda v} v_{xx}\right] \end{aligned} \tag{2.4.19}$$

in $\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)$, where

$$\gamma(\varphi) \stackrel{\text{def}}{=} \int_{\{y=0\}} (G_1 - G_2)\, \varphi dx \tag{2.4.20}$$

and where

$$\begin{aligned} G_1(x,y) &\stackrel{\text{def}}{=} g(x, y + u(x) + \lambda v(x))\sqrt{1 + (u'(x) + \lambda v'(x))^2}, \\ G_2(x,y) &\stackrel{\text{def}}{=} g(x, y + u(x))\sqrt{1 + (u'(x))^2}. \end{aligned} \tag{2.4.21}$$

Observe that

$$\|G_1 - G_2\|_{W^{1,q}\left(\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)\right)} \le c\lambda. \tag{2.4.22}$$

Now since

$$\operatorname{dist}\left(\partial\left(\Xi_{u+\lambda v}^{-1}(\Omega)\right), \partial\left(\Xi_u^{-1}(\Omega)\right)\right) \le c\lambda \tag{2.4.23}$$

and because of the Hölder continuity of $z^{u+\lambda v}$ and z^u, we conclude that

$$\|\tilde{z}^{u+\lambda v} - \tilde{z}^u\|_{C^0\left(\partial\left(\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)\right)\right)} \le c\lambda^\alpha. \tag{2.4.24}$$

Then (2.4.19), (2.4.22), and (2.4.24) imply that

$$\|\tilde{z}^{u+\lambda v} - \tilde{z}^u\|_{C^0\left(\overline{\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)}\right)} \le c\lambda^\alpha. \tag{2.4.25}$$

Then we have (set $A = (x,y)$)

$$\begin{aligned} & \left|z^{u+\lambda v}(A) - z^u(A)\right| \\ & \quad = \left|\tilde{z}^{u+\lambda v}\left(\Xi_{u+\lambda v}^{-1}(A)\right) - \tilde{z}^u\left(\Xi_u^{-1}(A)\right)\right| \\ & \quad \le \left|\tilde{z}^{u+\lambda v}\left(\Xi_{u+\lambda v}^{-1}(A)\right) - \tilde{z}^u\left(\Xi_{u+\lambda v}^{-1}(A)\right)\right| \\ & \qquad + \left|\tilde{z}^u\left(\Xi_{u+\lambda v}^{-1}(A)\right) - \tilde{z}^u\left(\Xi_u^{-1}(A)\right)\right| \\ & \quad \le c\lambda^\alpha + c\lambda^\alpha = c\lambda^\alpha. \end{aligned} \tag{2.4.26}$$

In (2.4.26), we also used the Hölder continuity of $\tilde{z}^u$. This completes the proof of the lemma.

COROLLARY 2.4.1.

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \int_{\Gamma_u} \left(z^{u+\lambda v} - z^u\right)^2 d\sigma = 0. \tag{2.4.27}$$

Proof. Take $\alpha > \frac{1}{2}$ in the lemma. Then

$$\frac{\left\|z^{u+\lambda v} - z^u\right\|_{C^0(\bar{\Omega})}^2}{\lambda} \le c\lambda^\beta,\ \beta = 2\alpha - 1 > 0. \tag{2.4.28}$$

Now we can proceed with the proof of the theorem. We compute the last term in (2.4.14).

$$\begin{aligned}
&\lim_{\lambda\downarrow 0}\frac{1}{2\lambda}\int_{\Gamma_u}\left((z^{u+\lambda v})^2-(z^u)^2\right)d\sigma\\
&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\int_{\Gamma_u}(z^{u+\lambda v}-z^u)z^u d\sigma+\lim_{\lambda\downarrow 0}\frac{1}{2\lambda}\int_{\Gamma_u}\left(z^{u+\lambda v}-z^u\right)^2 d\sigma\\
&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\int_{\Gamma_u}(z^{u+\lambda v}-z^u)z^u d\sigma=-\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\int_{\Omega}\nabla p^u\cdot\nabla(z^{u+\lambda v}-z^u)\\
&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\left(\int_{\Gamma_{u+\lambda v}}gp^u d\sigma-\int_{\Gamma_u}gp^u d\sigma\right)\\
&=\int_{\Gamma_u}\left((gp^{u,\mathrm{ext}})_y v^+-(gp^{u,\mathrm{int}})_y v^-\right)d\sigma+\int_{-1}^{1}gp^u\frac{u'v'}{\sqrt{1+u'^2}}dx. \qquad (2.4.29)
\end{aligned}$$

Now from (2.4.14) and (2.4.29) we conclude that Φ is directionally differentiable, and that (2.4.9) holds. Furthermore, if (2.4.10) holds, then

$$\begin{aligned}
&\Phi'(u;v)\\
&=\int_{-1}^{1}\left(z^u(z_y^{u,\mathrm{ext}}v^+-z_y^{u,\mathrm{int}}v^-)\sqrt{1+u'^2}+\frac{1}{2}(z^u)^2\frac{u'v'}{\sqrt{1+u'^2}}\right)dx\\
&\quad+\int_{\Gamma_u}\left((gp^{u,\mathrm{ext}})_y v^+-(gp^{u,\mathrm{int}})_y v^-\right)d\sigma+\int_{-1}^{1}gp^u\frac{u'v'}{\sqrt{1+u'^2}}dx\\
&\geq\int_{-1}^{1}\left(\tau-\left(\frac{u'}{\sqrt{1+u'^2}}\left(\frac{1}{2}(z^u)^2+gp^u\right)\right)'\right)vdx \qquad (2.4.30)
\end{aligned}$$

for all

$$\begin{aligned}
\tau\in\Big[&\left(z^u z_y^{u,\mathrm{int}}+(gp^{u,\mathrm{int}})_y\right)\sqrt{1+u'^2},\\
&\left(z^u z_y^{u,\mathrm{ext}}+(gp^{u,\mathrm{ext}})_y\right)\sqrt{1+u'^2}\Big]. \qquad (2.4.31)
\end{aligned}$$

This proves that Φ is subdifferentiable at u and that (2.4.11) holds. Similarly, one can consider superdifferentiability of Φ. So the theorem follows.

REMARK 2.4.1. *The previous suggests the numerical algorithm* (the steepest descent method) *for minimization of Φ, i.e., for the numerical solution of the relaxed free boundary problem:*

Choose $u_0\in U$. If $u_n\in U$ is already known, then u_{n+1} is determined as follows:

- *compute z^{u_n} as a solution of* (2.2.2);
- *compute p^{u_n} as a solution of* (2.4.4);
- *if* (2.4.10) *holds, compute an u_{n+1} such that*

$$u_{n+1}\in\left(u_n-\rho_n A^{-1}\left(\partial_*\Phi(u_n)\right)\right)\cap U,\ \rho_n>0, \qquad (2.4.32)$$

and if (2.4.12) *holds, compute an u_{n+1} such that*

$$u_{n+1}\in\left(u_n-\rho_n A^{-1}\left(\partial^*\Phi(u_n)\right)\right)\cap U,\ \rho_n>0. \qquad (2.4.33)$$

Here, A is the isomorphism between $H_0^3(-1,1)$ and its dual. More precisely, $\bar{u} = A^{-1}(l)$ means that $\bar{u}$ solves the following boundary value problem:

$$-\frac{d^6\bar{u}}{dx^6} = l \text{ in } (-1,1),$$
$$\bar{u}(-1) = \bar{u}'(-1) = \bar{u}''(-1) = \bar{u}(1) = \bar{u}'(1) = \bar{u}''(1) = 0. \tag{2.4.34}$$

If neither (2.4.10) *nor* (2.4.12) *holds, i.e., if Φ is neither convex nor concave at the point u_n, then it is more delicate to determine the steep(est) descent direction.*

3. Stokes flow.

3.1. Statement of the problem. The purpose of this section is to extend the previous results to the case of Stokes flow.

We consider a motionless body $\mathcal{B}$ in a viscous incompressible fluid moving in a bounded region Λ containing $\mathcal{B}$. The boundary of the region Λ will be denoted as $\partial\Lambda$. Fluid is moving at the velocity $\mathbf{h}$ at $\partial\Lambda$, and $\mathbf{h}$ is such that $\int_{\partial\Lambda} \mathbf{h}\cdot\mathbf{n}d\sigma = 0$, where $\mathbf{n}$ is the unit (exterior) normal to $\partial\Lambda$.

The boundary of the body $\partial\mathcal{B}$ consists of two disjoint and connected parts Σ and Γ, $\partial\mathcal{B} = \Sigma \cup \Gamma$. We shall suppose that Γ can be described as

$$\Gamma = \Gamma_u = \{(x_1, u(x_1)); -1 < x_1 < 1\} \tag{3.1.1}$$

for some function $u \in U$, where

$$U = \left\{u \in H_0^3(-1,1); 0 \le u(x_1) \le 1, -1 < x_1 < 1\right\}. \tag{3.1.2}$$

So, if we want to emphasize the dependence on $u \in U$ we shall write also $\mathcal{B} = \mathcal{B}_u$.

Denote by Ω_u the actual flow region $\Omega_u \stackrel{\text{def}}{=} \Lambda \setminus \bar{\mathcal{B}}_u$. Also, we assume that Σ is such that $\partial\mathcal{B}$ is sufficiently regular. Finally, denote,

$$\Omega = \{(x_1,x_2);\ -1 < x_1 < 1,\ 0 < x_2 < u(x_1)\} \cup \Omega_u \cup \Gamma_u, \tag{3.1.3}$$

so that $\Omega \setminus \bar{\Omega}_u = \{(x_1,x_2);\ -1 < x_1 < 1,\ 0 < x_2 < u(x_1)\}$.

Now, the velocity vector field of the fluid $\mathbf{w} = \mathbf{w}^u$, and the pressure p, are the solution of the Stokes system

$$-\nu\Delta\mathbf{w} + \nabla p = \mathbf{0} \text{ in } \Omega_u,\ \nabla\cdot\mathbf{w} = 0 \text{ in } \Omega_u,$$
$$\mathbf{w} = 0 \text{ in } \Gamma_u \cup \Sigma,\ \mathbf{w} = \mathbf{h} \text{ in } \partial\Lambda \tag{3.1.4}$$

We observe that the pressure p in (3.1.4) is determined only uniquely up to the additive constant.

The problem we propose is the following:

For given $\mathbf{g} = \mathbf{g}(x_1,x_2)$ such that (we will not always have to assume this much)

$$\mathbf{g} \in C^{1,1}(\Omega)^2, \tag{3.1.5}$$
$$\mathbf{g} = \mathbf{0} \text{ in } \Omega \cap \{|x_1| > 1\}, \tag{3.1.6}$$

find (if possible) $u \in U$ such that if $\mathbf{w}^u$ is the corresponding solution of (3.1.4), then also

$$-pn_j + \nu\left(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}\right)n_i = g_j \text{ in } \Gamma_u,\ j = 1,2, \tag{3.1.7}$$

where $\mathbf{n} = (n_i)_{i=1,2}$ is the unit normal exterior to Ω_u. We observe now that if (3.1.7) is to be satisfied in addition to (3.1.4) (and if (3.1.7) and (3.1.4) have a solution) then pressure p is determined *uniquely.* Also, we note that condition (3.1.7) means that fluid motion exhibits force distribution $\mathbf{g}$ on the boundary Γ_u. So, the problem we propose is to find a shape of the immersed body so that the prescribed force field is generated at the boundary.

We can simplify this problem right away. Let, as usual, $V = V(\Omega) \stackrel{\text{def}}{=} \{\mathbf{u} \in H_0^1(\Omega)^2;\ \nabla \cdot \mathbf{u} = 0\}$. We have the following lemma.

LEMMA 3.1.1. *If* $\mathbf{w} \in V \cap H^2(\Omega)^2$, *and* $\partial\Omega \in C^{0,1}$, *then*

$$\frac{\partial w_i}{\partial x_j} n_i = 0 \ on\ \partial\Omega. \tag{3.1.8}$$

Proof. Let $\varphi \in C^\infty(R^2)^2$ be an arbitrary function. We have

$$0 = \int_\Omega \frac{\partial w_i}{\partial x_i}\frac{\partial \varphi_j}{\partial x_j} = \int_\Omega \frac{\partial w_i}{\partial x_j}\frac{\partial \varphi_j}{\partial x_i} = \int_{\partial\Omega} \frac{\partial w_i}{\partial x_j} n_i \varphi_j. \tag{3.1.9}$$

So the lemma follows.

Lemma 3.1.1 implies that requesting (3.1.7) (in addition to (3.1.4)) is equivalent to requesting

$$-pn_j + \nu \frac{\partial w_j}{\partial x_i} n_i = g_j \text{ in } \Gamma_u,\ j = 1, 2. \tag{3.1.10}$$

Note that (3.1.10) is closely related to the equvalence of problems (3.2.2) and (3.2.3) (and hence, (3.2.4)).

3.2. Relaxation of the problem. Suppose that there exist a u, and a pair $(\mathbf{w}^u, p^u)$, a solution of (3.1.4), such that also (3.1.7) holds. So, we suppose existence for the free boundary problem (3.1.4) and (3.1.7). To refer to such an assumption we shall say that u is supposed to be an *exact shape.* Extend $\mathbf{w}^u$ from Ω_u to Ω as $\mathbf{z}^u$:

$$\mathbf{z}^u = \begin{cases} 0 & \text{on } \Omega \setminus \bar{\Omega_u}, \\ \mathbf{w}^u & \text{on } \Omega_u. \end{cases} \tag{3.2.1}$$

Lemma 3.2.1 follows.

LEMMA 3.2.1. *If u is an exact shape, then* $\mathbf{z}^u \in H^1(\Omega)^2$, *and it solves the Stokes system* (*with singular right-hand side*)

$$\frac{\nu}{2}\int_\Omega D(\mathbf{z}^u) : D(\varphi) = \xi_u(\varphi),\ \ \forall\varphi \in V, \tag{3.2.2}$$

i.e.,

$$\nu \int_\Omega \nabla \mathbf{z}^u : \nabla\varphi = \xi_u(\varphi),\ \ \forall\varphi \in V, \tag{3.2.3}$$

i.e.,

$$\begin{gathered} -\nu\Delta \mathbf{z}^u + \nabla p = \xi_u,\ \ in\ \mathcal{D}'(\Omega)^2, \\ \nabla \cdot \mathbf{z}^u = 0,\ \ a.e.\ in\ \Omega, \end{gathered} \tag{3.2.4}$$

where $\xi_u \in H^{-1}(\Omega)^2$ *is a signed vector measure given by*

$$\xi_u(\varphi) = \int_{\Gamma_u} \mathbf{g} \cdot \varphi d\sigma. \tag{3.2.5}$$

Here, $D(\mathbf{z}^u) : D(\varphi^u) = (\frac{\partial z_j}{\partial x_i} + \frac{\partial z_i}{\partial x_j})(\frac{\partial \varphi_j}{\partial x_i} + \frac{\partial \varphi_i}{\partial x_j})$, and throughout the paper the summation convention is assumed.

Proof of the Lemma. The proof is similar to the proof of Lemma 2.2.1; see also the proof of Lemma 3.2.2.

LEMMA 3.2.2. *Let* $u \in U$ *be given. Then, if* $\mathbf{z}^u$ *is a solution of* (3.2.2) *and if it happens that* $\mathbf{z}^u|_{\Gamma_u} = 0$, *then there exists pressure* p^u *such that* $(\mathbf{z}^u|_{\Omega_u}, p^u|_{\Omega_u})$ *is a solution of* (3.2.2) *and* (3.1.7), *i.e., u is an exact shape.*

Proof. Observe that any test function $\varphi \in V$ must satisfy $0 = \int_{\Omega\setminus\bar{\Omega}_u} \nabla \cdot \varphi = \int_{\partial(\Omega\setminus\bar{\Omega}_u)} \varphi \cdot \mathbf{n} d\sigma = \int_{\Gamma_u} \varphi \cdot \mathbf{n} d\sigma$. From (3.2.2) we see (here we assume the regularity of $\mathbf{z}^u|_{\Omega\setminus\bar{\Omega}_u}$ and $\mathbf{z}^u|_{\Omega_u}$, to be proved in the next section),

$$\begin{aligned}
\int_{\Gamma_u} \mathbf{g} \cdot \varphi d\sigma &= \frac{\nu}{2}\left(\int_{\Omega\setminus\bar{\Omega}_u} D(\mathbf{z}^u) : D(\varphi) + \int_{\Omega_u} D(\mathbf{z}^u) : D(\varphi)\right) \\
&= \int_{\Omega\setminus\bar{\Omega}_u} (-\nu\Delta\mathbf{z}^u) \cdot \varphi + \int_{\Omega_u} (-\nu\Delta\mathbf{z}^u) \cdot \varphi \\
&\quad + \int_{\partial(\Omega\setminus\bar{\Omega}_u)} \nu D(\mathbf{z}^u)\mathbf{n} \cdot \varphi d\sigma + \int_{\partial\Omega_u} \nu D(\mathbf{z}^u)\mathbf{n} \cdot \varphi d\sigma.
\end{aligned} \tag{3.2.6}$$

To fix ideas, let $\mathbf{n}$ be the unit normal exterior to Ω_u. Then (3.2.6) is equal to

$$\begin{aligned}
&\int_{\Omega\setminus\bar{\Omega}_u} (-\nu\Delta\mathbf{z}^u) \cdot \varphi + \int_{\Omega_u} (-\nu\Delta\mathbf{z}^u) \cdot \varphi \\
&\quad + \int_{\Gamma_u} \nu\left(D(\mathbf{z}^u)^{\text{ext}} - D(\mathbf{z}^u)^{\text{int}}\right)\mathbf{n} \cdot \varphi d\sigma
\end{aligned} \tag{3.2.7}$$

where $f^{\text{int}} \stackrel{\text{def}}{=} f|_{\Omega\setminus\bar{\Omega}_u}$ and $f^{\text{ext}} \stackrel{\text{def}}{=} f|_{\Omega_u}$.

We conclude that there exists $p^u \in L^2(\Omega)$ such that

$$-\nu\Delta\mathbf{z}^u + \nabla p^u = \mathbf{0} \text{ in } \Omega \setminus \bar{\Omega}_u, \tag{3.2.8}$$

$$-\nu\Delta\mathbf{z}^u + \nabla p^u = \mathbf{0} \text{ in } \Omega_u. \tag{3.2.9}$$

For such p we have

$$\begin{aligned}
\int_{\Gamma_u} \mathbf{g} \cdot \varphi d\sigma &= \int_{\Omega\setminus\bar{\Omega}_u} (-\nabla p^u) \cdot \varphi + \int_{\Omega_u} (-\nabla p^u) \cdot \varphi \\
&\quad + \int_{\Gamma_u} \nu\left(D(\mathbf{z}^u)^{\text{ext}} - D(\mathbf{z}^u)^{\text{int}}\right)\mathbf{n} \cdot \varphi d\sigma \\
&= \int_{\Omega} p^u \nabla \cdot \varphi - \int_{\partial(\Omega\setminus\bar{\Omega}_u)} p^u \varphi \cdot \mathbf{n} d\sigma - \int_{\partial\Omega_u} p^u \varphi \cdot \mathbf{n} d\sigma \\
&\quad + \int_{\Gamma_u} \nu\left(D(\mathbf{z}^u)^{\text{ext}} - D(\mathbf{z}^u)^{\text{int}}\right)\mathbf{n} \cdot \varphi d\sigma.
\end{aligned} \tag{3.2.10}$$

Let $[f]_{\text{int}}^{\text{ext}} \stackrel{\text{def}}{=} f^{\text{ext}}|_{\Gamma_u} - f^{\text{int}}|_{\Gamma_u}$ be the jump accross Γ_u. Then (3.2.10) implies that

$$\int_{\Gamma_u} g_j \varphi_j d\sigma = \int_{\Gamma_u} \left[-p^u n_j + \nu \left(\frac{\partial z_j^u}{\partial x_i} + \frac{\partial z_i^u}{\partial x_j} \right) n_i \right]_{\text{int}}^{\text{ext}} \varphi_j d\sigma \tag{3.2.11}$$

for all φ such that $\int_{\Gamma_u} \varphi_j n_j = 0$. This implies

$$\left[-p^u n_j + \nu \left(\frac{\partial z_j^u}{\partial x_i} + \frac{\partial z_i^u}{\partial x_j} \right) n_i \right]_{\text{int}}^{\text{ext}} - g_j = (\text{const}) n_j, \; j = 1, 2. \tag{3.2.12}$$

We note that (3.2.12) holds for any p^u such that (3.2.8) and (3.2.9) hold.

Now suppose that $\mathbf{z}^u|_{\Gamma_u} = 0$. Then $\mathbf{z}^u|_{\Omega \setminus \bar{\Omega}_u} = 0$, and without loss of generality we can choose $p^u|_{\Omega \setminus \bar{\Omega}_u} = 0$. Then (3.2.12) implies

$$\left. \left(-p^u n_j + \nu \left(\frac{\partial z_j^u}{\partial x_i} + \frac{\partial z_i^u}{\partial x_j} \right) n_i \right)^{\text{ext}} \right|_{\Gamma_u} - g_j = (\text{const}) n_j, \; j = 1, 2, \tag{3.2.13}$$

for any p^u satisfying (3.2.9). So, there exists p^u such that (3.2.9) holds and such that

$$\left. \left(-p^u n_j + \nu \left(\frac{\partial z_j^u}{\partial x_i} + \frac{\partial z_i^u}{\partial x_j} \right) n_i \right)^{\text{ext}} \right|_{\Gamma_u} - g_j = 0, \; j = 1, 2, \tag{3.2.14}$$

which is nothing but (3.1.7).

Lemma 3.2.2 motivates the following definition.

DEFINITION 3.2.1. *$u^* \in U$ is said to solve the* relaxed free boundary problem *if the corresponding $\mathbf{z}^u$ defined by* (3.2.2), *is such that if*

$$\Phi(u) = \frac{1}{2} \int_{\Gamma_u} |\mathbf{z}^u|^2 d\sigma \tag{3.2.15}$$

then

$$\Phi(u^*) = \min_{u \in U} \Phi(u). \tag{3.2.16}$$

3.3. The state equation. Let s be the segment

$$s = \{-1 \le x \le 1, y = 0\}, \tag{3.3.1}$$

and let

$$S_\epsilon = \{(x, y); \text{dist}((x, y), s) \le \epsilon\}. \tag{3.3.2}$$

Define

$$W_{s-\text{loc}}^{2,q}(\Omega) \stackrel{\text{def}}{=} \cap_{\epsilon > 0} W^{2,q}(\Omega \setminus S_\epsilon). \tag{3.3.3}$$

Now we have the next theorem.

THEOREM 3.3.1. *Let q be such that $1 < q < \infty$, and $\partial\Omega \in C^2$. Let $u \in U$. Then the state equation* (3.2.2) *has a unique weak solution $\mathbf{z}^u$, and*

$$\mathbf{z}^u \in V \cap C^\infty(\Omega \setminus \Gamma_u)^2. \tag{3.3.4}$$

More importantly, the following regularity results hold:

(a) L^p-estimate: *If* $\mathbf{g} \in W^{1,q}(\Omega)^2$, *then*

$$\|\mathbf{z}^u\|_{W^{1,q}(\Omega)^2} + \|p^u\|_{L^q(\Omega)/R} \leq c\left(1 + \|u\|_{C^{0,1}(-1,1)}\right) \cdot \left(\|\mathbf{g}\|_{W^{1,q}(\Omega)^2} + \|\mathbf{h}\|_{W^{1-\frac{1}{q},q}(\partial\Omega)^2}\right). \tag{3.3.5}$$

(b) maximum modulus estimate: *If, moreover,* $q > 2$, *then*

$$\|\mathbf{z}^u\|_{L^\infty(\Omega)^2} \leq c\left(1 + \|u\|_{C^{0,1}(-1,1)}\right) \left(\|\mathbf{g}\|_{W^{1,q}(\Omega)^2} + \|\mathbf{h}\|_{L^\infty(\partial\Omega)^2}\right). \tag{3.3.6}$$

(c) *If* $\mathbf{g} \in W^{2,q}(\Omega)^2$, *and* (3.1.6) *holds, then*

$$\mathbf{z}^{u,\text{ext}} \in W^{2,q}(\Omega_u)^2, \ \mathbf{z}^{u,\text{int}} \in W^{2,q}_{s-\text{loc}}(\Omega \setminus \bar{\Omega}_u)^2 \tag{3.3.7}$$

and the a priori estimates

$$\|\mathbf{z}^{u,\text{ext}}\|_{W^{2,q}(\Omega_u)^2} + \|p^{u,\text{ext}}\|_{W^{1,q}(\Omega)/R} \leq c\left(\|u\|_{H^3(-1,1)}, \|\mathbf{g}\|_{W^{2,q}(\Omega)^2}, \|\mathbf{h}\|_{W^{2-\frac{1}{q},q}(\partial\Omega)^2}\right), \tag{3.3.8}$$

and

$$\|\mathbf{z}^{u,\text{int}}\|_{W^{2,q}((\Omega\setminus\Omega_u)\cap\{y>\epsilon\})^2} + \|p^{u,\text{int}}\|_{W^{1,q}((\Omega\setminus\Omega_u)\cap\{y>\epsilon\})/R} \leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|\mathbf{g}\|_{W^{2,q}(\Omega)^2}, \|\mathbf{h}\|_{W^{2-\frac{1}{q},q}(\partial\Omega)^2}\right) \tag{3.3.9}$$

hold.

Proof. The interior regularity $\mathbf{z}^u \in C^\infty(\Omega \setminus \Gamma_u)^2$ follows easily (see, e.g., [6]). The proof of (3.3.5) is similar to the proof of (2.3.4). One has to use (see [5]) the following L^q-estimate for the Stokes problem:

$$\|\mathbf{z}^u\|_{W^{1,q}(\Omega)^2} + \|p^u\|_{L^q(\Omega)/R} \leq c\left\{\|F\|_{W^{-1,q}(\Omega)^2} + \|\mathbf{h}\|_{W^{1-\frac{1}{q},q}(\partial\Omega)^2}\right\}, \tag{3.3.10}$$

where F is the right-hand side; in our case $F(\varphi) = \int_{\Gamma_u} \mathbf{g} \cdot \varphi d\sigma$.

We prove now (3.3.6). We use the following important result (see [13]) for biharmonic functions:

If $\Delta^2\varphi = 0$ in Ω, then

$$\|\nabla\varphi\|_{L^\infty(\Omega)^2} \leq c\|\nabla\varphi\|_{L^\infty(\partial\Omega)^2}. \tag{3.3.11}$$

Let $\mathbf{w}$ solve the homogeneous Stokes problem

$$-\nu\Delta\mathbf{w} + \nabla p = 0 \ \text{ in } \ \Omega, \tag{3.3.12}$$

$$\nabla \cdot \mathbf{w} = 0 \ \text{ in } \ \Omega, \tag{3.3.13}$$

$$\mathbf{w} = \mathbf{h} \ \text{ on } \ \partial\Omega, \tag{3.3.14}$$

where $\mathbf{h}$ is such that $\int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n} d\sigma = 0$. It is well known (see, e.g., [10]) that (3.3.13) and (3.3.14) imply the existence of φ such that $\mathbf{w} = \mathbf{curl}\varphi$. Here $\mathbf{curl}\varphi = \langle \frac{\partial\varphi}{\partial y}, -\frac{\partial\varphi}{\partial x} \rangle$.

Also let $\operatorname{curl}\mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$. Now, since $\operatorname{curl}\,\mathbf{curl}\varphi = -\Delta\varphi$, and since $\operatorname{curl}\nabla p = 0$, we conclude from (3.3.12) that $\Delta^2\varphi = 0$, and hence (3.3.11) follows. But then we have

$$\begin{aligned}
&\|\mathbf{w}\|_{L^\infty(\Omega)^2} = \|\mathbf{curl}\varphi\|_{L^\infty(\Omega)^2} = \|\nabla\varphi\|_{L^\infty(\Omega)^2} \\
&\le c\|\nabla\varphi\|_{L^\infty(\partial\Omega)^2} = c\|\mathbf{curl}\varphi\|_{L^\infty(\partial\Omega)^2} = c\|\mathbf{h}\|_{L^\infty(\partial\Omega)^2}.
\end{aligned} \tag{3.3.15}$$

Now, (3.3.6) follows by linearity from (3.3.5) and (3.3.15) and by the imbedding theorem.

As in §2, define $\tilde{\mathbf{z}}$, $\tilde{p}$, $\tilde{\mathbf{g}}$ and $\tilde{\varphi}$ by $\tilde{\mathbf{z}}(x,y) \stackrel{\text{def}}{=} \mathbf{z}^u(x, y+u(x)), \tilde{p}(x,y) \stackrel{\text{def}}{=} p^u(x, y+u(x)), \tilde{\mathbf{g}}(x,y) \stackrel{\text{def}}{=} \mathbf{g}(x, y+u(x))\sqrt{1+u'^2(x)}, \tilde{\varphi}(x,y) \stackrel{\text{def}}{=} \varphi(x, y+u(x))$. Define operator L by $Lv = \Delta v + v_{yy}(u_x)^2 - 2v_{xy}u_x - v_y u_{xx}$, and $\tilde{\nabla}$ by $\tilde{\nabla}\tilde{p} = \langle \tilde{p}_x - \tilde{p}_y u_x, \tilde{p}_y\rangle$. Then, as before, L is uniformly elliptic.

Let, also, Ξ_u be the map with the image Ω given by the formula $\Xi_u(x,y) = (x, y+u(x))$. Then, $(-\nu\Delta\mathbf{z}^u + \nabla p^u)\circ\Xi_u = -\nu L\tilde{\mathbf{z}} + \tilde{\nabla}\tilde{p}$, and since $|\det D\Xi_u| = 1$ (here $D\Xi_u$ is the gradient matrix of the map Ξ_u so that $|\det D\Xi_u|$ is the Jacobian)

$$\left(-\nu L\tilde{\mathbf{z}} + \tilde{\nabla}\tilde{p}\right)(\tilde{\varphi}) = (-\nu\Delta\mathbf{z}^u + \nabla p^u)(\varphi). \tag{3.3.16}$$

Hence

$$\begin{aligned}
\left(-\nu L\tilde{\mathbf{z}} + \tilde{\nabla}\tilde{p}\right)(\tilde{\varphi}) &= \int_{\Gamma_u} \mathbf{g}\cdot\varphi\, d\sigma \\
&= \int \mathbf{g}(x,u(x))\cdot\varphi(x,u(x))\sqrt{1+u'^2(x)}\,dx \\
&= \int_{\{y=0\}} \tilde{\mathbf{g}}\cdot\tilde{\varphi}\,dx \stackrel{\text{def}}{=} \tilde{\xi}(\tilde{\varphi}).
\end{aligned} \tag{3.3.17}$$

So,

$$-\nu L\tilde{\mathbf{z}} + \tilde{\nabla}\tilde{p} = \tilde{\xi} \tag{3.3.18}$$

in the sense of distributions. Then from (3.3.18) we get

$$\left(-\nu L\tilde{\mathbf{z}} + \tilde{\nabla}\tilde{p}\right)\left(\delta^1_{-h}\tilde{\varphi}\right) = \tilde{\xi}\left(\delta^1_{-h}\tilde{\varphi}\right). \tag{3.3.19}$$

We have

$$\begin{aligned}
\tilde{\xi}\left(\delta^1_{-h}\tilde{\varphi}\right) = \int_{\{y=0\}} \tilde{\mathbf{g}}\cdot\delta^1_{-h}\tilde{\varphi}\,dx &= -\int_{\{y=0\}} \left(\delta^1_h\tilde{\mathbf{g}}\right)\cdot\tilde{\varphi}\,dx \\
&\longrightarrow -\int_{\{y=0\}} \tilde{\mathbf{g}}_x\cdot\tilde{\varphi}\,dx \stackrel{\text{def}}{=} -\tilde{\xi}_x(\tilde{\varphi}),
\end{aligned} \tag{3.3.20}$$

as $h \longrightarrow 0$. We conclude that $(-\nu L\tilde{\mathbf{z}} + \tilde{\nabla}\tilde{p})_x(\tilde{\varphi}) = \tilde{\xi}_x(\tilde{\varphi})$, and hence

$$\begin{aligned}
&-\nu L_1\tilde{\mathbf{z}}_x + \tilde{\nabla}\tilde{p}_x \\
&= \tilde{\xi}_x + \nu\left(\tilde{\mathbf{z}}_{yy}2u_x u_{xx} - \tilde{\mathbf{z}}_y u_{xxx}\right) + \langle \tilde{p}_y u_{xx}, 0\rangle,
\end{aligned} \tag{3.3.21}$$

where $L_1 w = \Delta w + (u_x)^2 w_{yy} - 2u_x w_{xy} - 3u_{xx}w_y$. The rest of the proof is similar to the proof of (2.3.8).

Again, in order to claim existence of a minimizer, i.e., existence of a solution of the relaxed problem, one needs some kind of compactness. So let

$$U_b = \{u \in U; \|u\|_{H^3(-1,1)} \leq b\}, \tag{3.3.22}$$

where b is some prescribed positive constant.

PROPOSITION 3.3.1. *Let $\mathbf{g} \in W^{1,q}(\Omega)^2$, for some $q > 2$. Then, there exists an $u^* \in U_b$ such that*

$$\Phi(u^*) = \min_{u \in U_b} \Phi(u). \tag{3.3.23}$$

Proof. The proof is similar to the proof of Proposition 2.3.1.

3.4. Differentiability properties of the variational functional Φ. Our goal is to derive information about the *multivalued* generalized gradient of Φ.

Define the adjoint variable ζ^u, as a solution of the (adjoint) equation

$$\begin{aligned} -\nu\Delta\zeta^u + \nabla q = \eta_u \ \text{ in } \ \mathcal{D}'(\Omega)^2, \ \nabla\cdot\zeta^u = 0 \ \text{ a.e. in } \ \Omega, \\ \zeta^u = 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{3.4.1}$$

where $\eta_u \in H^{-1}(\Omega)$ is a vector (signed) measure given by

$$\eta_u(\varphi) = \int_{\Gamma_u} \mathbf{z}^u \cdot \varphi d\sigma. \tag{3.4.2}$$

Obviously, (3.4.1) is the same type of equation as (3.2.2).

In this section, as before, $\mathbf{z}^{u,\text{ext}} = \mathbf{z}^u|_{\Omega_u}$ and $\mathbf{z}^{u,\text{int}} = \mathbf{z}^u|_{\Omega\setminus\Omega_u}$; also, below we shall use the notation $\zeta^{u,\text{ext}} = \zeta^u|_{\Omega_u}$ and $\zeta^{u,\text{int}} = \zeta^u|_{\Omega\setminus\Omega_u}$.

LEMMA 3.4.1. *Let $\mathbf{g} \in W^{2,q}(\Omega)^2$, for some $q > 1$. Then*

$$\zeta^{u,\text{ext}} \in W^{2,q}(\Omega_u)^2, \ \zeta^{u,\text{int}} \in W^{2,q}_{s-\text{loc}}(\Omega\setminus\bar{\Omega}_u)^2. \tag{3.4.3}$$

and the a priori estimates

$$\|\zeta^{u,\text{ext}}\|_{W^{2,q}(\Omega_u)^2} \leq c\left(\|u\|_{H^3(-1,1)}, \|\mathbf{g}\|_{W^{2,q}(\Omega)^2}, \|\mathbf{h}\|_{W^{2-\frac{1}{q},q}(\Omega)^2}\right), \tag{3.4.4}$$

and

$$\begin{aligned} &\| \zeta^{u,\text{int}}\|_{W^{2,q}(\Omega\setminus S_\epsilon)} \\ &\leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right) \end{aligned} \tag{3.4.5}$$

hold.

Proof. The proof is the same as the proof of Lemma 2.4.1.

We have the next theorem.

THEOREM 3.4.1. *Assume* (3.1.5) *and* (3.1.6). *Then Φ is directionally differentiable at any $u \in U$ such that $u(x) > 0$ for $-1 < x < 1$, and*

$$\begin{aligned} &\Phi'(u; v) \\ &= \int_{-1}^{1} \left(\mathbf{z}^u \cdot (\mathbf{z}_y^{u,\text{ext}} v^+ - \mathbf{z}_y^{u,\text{int}} v^-)\sqrt{1+u'^2} + \frac{1}{2}|\mathbf{z}^u|^2 \frac{u'v'}{\sqrt{1+u'^2}}\right) dx \\ &\quad + \int_{\Gamma_u} \left((\mathbf{g}\cdot\zeta^{u,\text{ext}})_y v^+ - (\mathbf{g}\cdot\zeta^{u,\text{int}})_y v^-\right) d\sigma \\ &\quad + \int_{-1}^{1} \mathbf{g}\cdot\zeta^u \frac{u'v'}{\sqrt{1+u'^2}} dx. \end{aligned} \tag{3.4.6}$$

Moreover, if

$$\mathbf{z}^u \cdot \mathbf{z}_y^{u,\mathrm{int}} + (\mathbf{g} \cdot \zeta^{u,\mathrm{int}})_y \leq \mathbf{z}^u \cdot \mathbf{z}_y^{u,\mathrm{ext}} + (\mathbf{g} \cdot \zeta^{u,\mathrm{ext}})_y \quad a.e.\ in\ (-1,1), \tag{3.4.7}$$

then Φ is subdifferentiable at u and

$$\begin{aligned} &\partial_*\Phi(u) \\ &= \left[\left(\mathbf{z}^u \cdot \mathbf{z}_y^{u,\mathrm{int}} + (\mathbf{g} \cdot \zeta^{u,\mathrm{int}})_y\right)\sqrt{1+u'^2}, \left(\mathbf{z}^u \cdot \mathbf{z}_y^{u,\mathrm{ext}} + (\mathbf{g} \cdot \zeta^{u,\mathrm{ext}})_y\right)\sqrt{1+u'^2}\right] \\ &\quad - \left(\frac{u'}{\sqrt{1+u'^2}}\left(\frac{1}{2}|\mathbf{z}^u|^2 + \mathbf{g} \cdot \zeta^u\right)\right)' \\ &\overset{\mathrm{def}}{=} [l\partial_*\Phi(u), r\partial_*\Phi(u)] \subset L^\infty(-1,1). \end{aligned} \tag{3.4.8}$$

On the other hand, if

$$\mathbf{z}^u \cdot \mathbf{z}_y^{u,\mathrm{int}} + (\mathbf{g} \cdot \zeta^{u,\mathrm{int}})_y \geq \mathbf{z}^u \cdot \mathbf{z}_y^{u,\mathrm{ext}} + (\mathbf{g} \cdot \zeta^{u,\mathrm{ext}})_y \quad a.e.\ in\ (-1,1), \tag{3.4.9}$$

then Φ is superdifferentiable at u and

$$\begin{aligned} &\partial^*\Phi(u) \\ &= \left[\left(\mathbf{z}^u \cdot \mathbf{z}_y^{u,\mathrm{ext}} + (\mathbf{g} \cdot \zeta^{u,\mathrm{ext}})_y\right)\sqrt{1+u'^2}, \left(\mathbf{z}^u \cdot \mathbf{z}_y^{u,\mathrm{int}} + (\mathbf{g} \cdot \zeta^{u,\mathrm{int}})_y\right)\sqrt{1+u'^2}\right] \\ &\quad - \left(\frac{u'}{\sqrt{1+u'^2}}\left(\frac{1}{2}|\mathbf{z}^u|^2 + \mathbf{g} \cdot \zeta^u\right)\right)' \\ &\overset{\mathrm{def}}{=} [l\partial^*\Phi(u), r\partial^*\Phi(u)] \subset L^\infty(-1,1). \end{aligned} \tag{3.4.10}$$

Proof. As before, we compute

$$\begin{aligned} \Phi'(u;v) &= \lim_{\lambda\downarrow 0} \frac{\Phi(u+\lambda v) - \Phi(u)}{\lambda} \\ &= \lim_{\lambda\downarrow 0} \frac{1}{2\lambda}\left(\int_{\Gamma_{u+\lambda v}} |\mathbf{z}^{u+\lambda v}|^2 d\sigma - \int_{\Gamma_u} |\mathbf{z}^u|^2 d\sigma\right) \\ &= \int_{-1}^{1}\left(\mathbf{z}^u \cdot (\mathbf{z}_y^{u,\mathrm{ext}} v^+ - z_y^{u,\mathrm{int}} v^-)\sqrt{1+u'^2} + \frac{1}{2}|\mathbf{z}^u|^2 \frac{u'v'}{\sqrt{1+u'^2}}\right) dx \\ &\quad + \lim_{\lambda\downarrow 0} \frac{1}{2\lambda}\int_{\Gamma_u} \left(|\mathbf{z}^{u+\lambda v}|^2 - |\mathbf{z}^u|^2\right) d\sigma. \end{aligned} \tag{3.4.11}$$

LEMMA 3.4.2. *Under previous assumptions on u and v, and for any $\alpha < 1$, the following estimate holds*

$$\|\mathbf{z}^{u+\lambda v} - \mathbf{z}^u\|_{C^0(\bar{\Omega})^2} \leq c\lambda^\alpha. \tag{3.4.12}$$

Proof. Let, as before, Ξ_u be the map with the image Ω given by the formula $\Xi_u(x,y) = (x, y + u(x))$. Then $\Xi_u^{-1}(x,y) = (x, y - u(x))$, and (set $A = (x,y)$) $\operatorname{dist}\left(\Xi_{u+\lambda v}^{-1}(A) - \Xi_u^{-1}(A)\right) \leq c\lambda$. Now consider $\tilde{\mathbf{z}}^{u+\lambda v}, \tilde{\mathbf{z}}^u, \tilde{p}^{u+\lambda v}, \tilde{p}^u$ defined as $\tilde{\mathbf{z}}^{u+\lambda v} = \mathbf{z}^{u+\lambda v} \circ \Xi_{u+\lambda v}, \tilde{\mathbf{z}}^u = \mathbf{z}^u \circ \Xi_u, \tilde{p}^{u+\lambda v} = p^{u+\lambda v} \circ \Xi_{u+\lambda v}, \tilde{p}^u = p^u \circ \Xi_u$, and operators L_u

and $L_{u+\lambda v}$ defined by

$$L_u w = \Delta w + w_{yy}(u_x)^2 - 2w_{xy}u_x - w_y u_{xx},$$

$$\begin{aligned} L_{u+\lambda v} w &= \Delta w + w_{yy}(u_x + \lambda v_x)^2 - 2w_{xy}(u_x + \lambda v_x) - w_y(u_{xx} + \lambda v_x) \\ &= L_u w + \lambda \left[w_{yy}(2u_x v_x + \lambda v_x^2) - 2w_{xy}v_x - w_y v_{xx} \right], \end{aligned} \tag{3.4.13}$$

and

$$\tilde{\nabla}_u w = \langle w_x - w_y u_x, w_y \rangle,$$

$$\tilde{\nabla}_{u+\lambda v} w = \langle w_x - w_y(u + \lambda v)_x, w_y \rangle = \tilde{\nabla}_u w - \langle \lambda v_x, 0 \rangle. \tag{3.4.14}$$

Then $\tilde{\mathbf{z}}^{u+\lambda v} - \tilde{\mathbf{z}}^u$ and corresponding $\tilde{p}^{u+\lambda v} - \tilde{p}^u$ satisfy the equation

$$\begin{aligned} -\nu L_u \left(\tilde{\mathbf{z}}^{u+\lambda v} - \tilde{\mathbf{z}}^u \right) + \tilde{\nabla}_u \left(\tilde{p}^{u+\lambda v} - \tilde{p}^u \right) \\ = \gamma + \langle \lambda v_x, 0 \rangle \\ + \nu \lambda \left[\tilde{\mathbf{z}}_{yy}^{u+\lambda v}(2u_x v_x + \lambda v_x^2) - 2\tilde{\mathbf{z}}_{xy}^{u+\lambda v} v_x - \tilde{\mathbf{z}}_y^{u+\lambda v} v_{xx} \right] \end{aligned} \tag{3.4.15}$$

in $\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)$, where

$$\gamma(\varphi) \stackrel{\text{def}}{=} \int_{\{y=0\}} (\mathbf{G}_1 - \mathbf{G}_2)\, \varphi dx \tag{3.4.16}$$

and where

$$\mathbf{G}_1(x,y) \stackrel{\text{def}}{=} \mathbf{g}(x, y + u(x) + \lambda v(x)) \sqrt{1 + (u'(x) + \lambda v'(x))^2}, \tag{3.4.17}$$

$$\mathbf{G}_2(x,y) \stackrel{\text{def}}{=} \mathbf{g}(x, y + u(x)) \sqrt{1 + (u'(x))^2}. \tag{3.4.18}$$

Observe that

$$\|\mathbf{G}_1 - \mathbf{G}_2\|_{W^{1,q}(\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega))^2} \le c\lambda. \tag{3.4.19}$$

Now since dist $\left(\partial\left(\Xi_{u+\lambda v}^{-1}(\Omega)\right), \partial\left(\Xi_u^{-1}(\Omega)\right)\right) \le c\lambda$ and because of the Hölder continuity of $\mathbf{z}^{u+\lambda v}$ and $\mathbf{z}^u$, we conclude that

$$\|\tilde{\mathbf{z}}^{u+\lambda v} - \tilde{\mathbf{z}}^u\|_{C^0(\partial(\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)))^2} \le c\lambda^\alpha. \tag{3.4.20}$$

Then (3.3.6), (3.4.19), and (3.4.20) imply that

$$\|\tilde{\mathbf{z}}^{u+\lambda v} - \tilde{\mathbf{z}}^u\|_{C^0\left(\overline{\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)}\right)^2} \le c\lambda^\alpha. \tag{3.4.21}$$

Then we have (set $A = (x,y)$)

$$\begin{aligned} &\left| \mathbf{z}^{u+\lambda v}(A) - \mathbf{z}^u(A) \right| \\ &\quad = \left| \tilde{\mathbf{z}}^{u+\lambda v}\left(\Xi_{u+\lambda v}^{-1}(A)\right) - \tilde{\mathbf{z}}^u\left(\Xi_u^{-1}(A)\right) \right| \\ &\quad \le \left| \tilde{\mathbf{z}}^{u+\lambda v}\left(\Xi_{u+\lambda v}^{-1}(A)\right) - \tilde{\mathbf{z}}^u\left(\Xi_{u+\lambda v}^{-1}(A)\right) \right| \\ &\qquad + \left| \tilde{\mathbf{z}}^u\left(\Xi_{u+\lambda v}^{-1}(A)\right) - \tilde{\mathbf{z}}^u\left(\Xi_u^{-1}(A)\right) \right| \\ &\qquad\quad c\lambda^\alpha + c\lambda^\alpha = c\lambda^\alpha. \end{aligned} \tag{3.4.22}$$

In (3.4.22) we also used the Hölder continuity of $\tilde{\mathbf{z}}^u$. This completes the proof of the lemma.

COROLLARY 3.4.1.

$$\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\int_{\Gamma_u}\left|\mathbf{z}^{u+\lambda v}-\mathbf{z}^u\right|^2 d\sigma=0. \tag{3.4.23}$$

Now, we can proceed with the proof of the theorem. We compute the last term in (3.4.11).

$$\begin{aligned}
&\lim_{\lambda\downarrow 0}\frac{1}{2\lambda}\int_{\Gamma_u}\left(|\mathbf{z}^{u+\lambda v}|^2-|\mathbf{z}^u|^2\right)d\sigma\\
&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\int_{\Gamma_u}(\mathbf{z}^{u+\lambda v}-\mathbf{z}^u)\cdot\mathbf{z}^u d\sigma+\lim_{\lambda\downarrow 0}\frac{1}{2\lambda}\int_{\Gamma_u}\left|\mathbf{z}^{u+\lambda v}-\mathbf{z}^u\right|^2 d\sigma\\
&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\int_{\Gamma_u}(\mathbf{z}^{u+\lambda v}-\mathbf{z}^u)\cdot\mathbf{z}^u d\sigma=\lim_{\lambda\downarrow 0}\frac{\nu}{\lambda}\int_{\Omega}\nabla\zeta^u:\nabla(\mathbf{z}^{u+\lambda v}-\mathbf{z}^u)\\
&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\left(\int_{\Gamma_{u+\lambda v}}\mathbf{g}\cdot\zeta^u d\sigma-\int_{\Gamma_u}\mathbf{g}\cdot\zeta^u d\sigma\right)\\
&=\int_{\Gamma_u}\left((\mathbf{g}\cdot\zeta^{u,\mathrm{ext}})_y v^+-(\mathbf{g}\cdot\zeta^{u,\mathrm{int}})_y v^-\right)d\sigma\\
&\quad+\int_{-1}^{1}\mathbf{g}\cdot\zeta^u\frac{u'v'}{\sqrt{1+u'^2}}dx.
\end{aligned} \tag{3.4.24}$$

Now from (3.4.11) and (3.4.24) we conclude that Φ is directionally differentiable, and that (3.4.6) holds. Furthermore, if (3.4.7) holds, then

$$\begin{aligned}
&\Phi'(u;v)\\
&=\int_{-1}^{1}\left(\mathbf{z}^u\cdot(\mathbf{z}_y^{u,\mathrm{ext}}v^+-\mathbf{z}_y^{u,\mathrm{int}}v^-)\sqrt{1+u'^2}+\frac{1}{2}|\mathbf{z}^u|^2\frac{u'v'}{\sqrt{1+u'^2}}\right)dx\\
&\quad+\int_{\Gamma_u}\left((\mathbf{g}\cdot\zeta^{u,\mathrm{ext}})_y v^+-(\mathbf{g}\cdot\zeta^{u,\mathrm{int}})_y v^-\right)d\sigma+\int_{-1}^{1}\mathbf{g}\cdot\zeta^u\frac{u'v'}{\sqrt{1+u'^2}}dx\\
&\geq\int_{-1}^{1}\left(\tau-\left(\frac{u'}{\sqrt{1+u'^2}}\left(\frac{1}{2}|\mathbf{z}^u|^2+\mathbf{g}\cdot\zeta^u\right)\right)'\right)vdx
\end{aligned} \tag{3.4.25}$$

for all

$$\begin{aligned}
\tau\in\Big[&\left(\mathbf{z}^u\cdot\mathbf{z}_y^{u,\mathrm{int}}+(\mathbf{g}\cdot\zeta^{u,\mathrm{int}})_y\right)\sqrt{1+u'^2},\\
&\left(\mathbf{z}^u\cdot\mathbf{z}_y^{u,\mathrm{ext}}+(\mathbf{g}\cdot\zeta^{u,\mathrm{ext}})_y\right)\sqrt{1+u'^2}\Big].
\end{aligned} \tag{3.4.26}$$

This proves that Φ is subdifferentiable at u and that (3.4.8) holds. Similarly, one can consider superdifferentiability of Φ. So the theorem follows.

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REFERENCES

[1] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] H. W. ALT AND L. A. CAFFARELLI, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math., 105 (1981), pp. 105–144.
[3] H. W. ALT, L. A. CAFFARELLI, AND A. FRIEDMAN, *A free boundary problem for quasi-linear elliptic equations*, Ann. Scuola Norm. Sup. Pisa, 11 (1984), pp. 1–44.
[4] V. BARBU AND S. STOJANOVIC, *A variational approach to a free boundary problem arising in electrophotography*, Numer. Funct. Anal. Optim., 14 (1993), pp. 1–14.
[5] L. CATTABRIGA, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rendiconti Sem. Mat. Univ. Padova, 31 (1961), pp. 308–340.
[6] P. CONSTANTIN AND C. FOIAS, *Navier-Stokes Equations*, The University of Chicago Press, Chicago, IL, 1990.
[7] L. C. EVANS AND R. F. GARIEPY, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
[8] A. FRIEDMAN, *Variational Principles and Free-Boundary Problems*, Wiley-Interscience, New York, 1982.
[9] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.
[10] V. GIRAULT AND P.-A. RAVIART, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1986.
[11] B. KAWOHL, *Rearrangements and Convexity of Level Sets in PDE*, LNM #1150, Springer-Verlag, Berlin, 1985.
[12] J. NEČAS, *Les Methodes Directes en Theorie des Equations Elliptiques*, Masson, Paris, 1967.
[13] J. PIPHER, manuscript.
[14] S. STOJANOVIC, *Parallel computations for variational free boundary problem modeling injection of fluid from a slot into a stream*, in Theoretical Aspects of Industrial Design, D.A. Field and V. Komkov, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.
[15] ———, *Nonsmooth Analysis and Shape Optimization in a Flow Problem*, Proceedings of the 31st IEEE Conf. on Decision and Control, pp. 3117–3118, 1992; IMA Preprint Series 1046.
[16] G. M. TROIANIELLO, *Elliptic Differential Equations and Obstacle Problems*, Plenum Press, New York, 1987.